Practice Final Measure and Integration I 12015-16

1. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$.

   (a) Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f\, d\mu \leq \int_E f\, d\nu$.

   (b) Prove that if ν is a finite measure, then $\mathcal{L}^2(\nu) \subseteq \mathcal{L}^1(\mu)$.

2. Consider the measure space $((0,1], \mathcal{B}((0,1]), \lambda)$, where $\mathcal{B}((0,1])$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to the interval $(0,1]$. Determine the value of

$$\lim_{n\to\infty} \int_{(0,1]} e^{1/x}(1+n^2x)^{-1} \sin(ne^{-1/x}\, d\lambda(x).$$

3. Let $(X, \mathcal{F}, \mu)$ be a **finite** measure space. Assume $f \in \mathcal{L}^2(\mu)$ satisfies $0 < ||f||_2 < \infty$, and let $A = \{x \in X : f(x) \neq 0\}$. Show that

$$\mu(A) \geq \frac{(\int f\, d\mu)^2}{\int f^2\, d\mu}.$$

4. Let $E = \{(x,y) : y < x < 1, , 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x,y) = x^{-3/2} \cos(\frac{\pi y}{2x})$.

   (a) Show that f is $\lambda \times \lambda$ integrable on E.

   (b) Define $F : (0,1) \to \mathbb{R}$ by $F(y) = \int_{(y,1)} x^{-3/2} \cos(\frac{\pi y}{2x})\, d\lambda(x)$. Determine the value of

$$\int F(y)\, d\lambda(y).$$

5. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and (f_j) a uniformly integrable sequence of measurable functions. Define $F_k = \sup_{1\leq j\leq k} |f_j|$ for $k \geq 1$.

   (a) Show that for any $w \in \mathcal{M}^+(\mathcal{A})$,

$$\int_{\{F_k > w\}} F_k\, d\mu \leq \sum_{j=1}^{k} \int_{\{|f_j|>w\}} |f_j|\, d\mu.$$

(b) Show that for every $\epsilon > 0$, there exists a $w_\epsilon \in \mathcal{L}^1_+(\mu)$ such that for all $k \geq 1$

$$\int_X F_k \, d\mu \leq \int_X w_\epsilon \, d\mu + k\epsilon.$$

(c) Show that

$$\lim_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = 0.$$

6. Suppose μ and ν are finite measures on the measurable space $(X, \mathcal{A})$ which have the same null sets. Show that there exists a measurable function f such that $0 < f < \infty$ μ a.e. and ν a.e. and for all $A \in \mathcal{A}$ one has

$$\nu(A) = \int_A f \, d\mu \quad \text{and} \quad \mu(A) = \int_A \frac{1}{f} \, d\nu.$$