Measure and Integration: Solutions Practice Final II 2015-16

(1) Consider a measure space $(X, \mathcal{A}, \mu)$, and let $(f_n)_n$ be a sequence in $\mathcal{L}^2(\mu)$ which is bounded in the $\mathcal{L}^2$ norm, i.e. there exists a constant $C > 0$ such that $||f_n||_2 < C$ for all $n \geq 1$.

(a) Prove that $\sum_{n=1}^{\infty} (\frac{f_n}{n})^2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$.

(b) Prove that $\lim_{n\to\infty} \frac{f_n}{n} = 0$ μ a.e.

Proof (a): First observe that

$$\sum_{n=1}^{\infty} ||\frac{f_n}{n}||_2^2 = \sum_{n=1}^{\infty} \frac{||f_n||_2^2}{n^2} \leq \sum_{n=1}^{\infty} \frac{C^2}{n^2} < \infty.$$

Now, by Beppo-Levi and the above, we have

$$\int \sum_{n=1}^{\infty} (\frac{f_n}{n})^2 \, d\mu = \sum_{n=1}^{\infty} \int (\frac{f_n}{n})^2 \, d\mu = \sum_{n=1}^{\infty} ||\frac{f_n}{n}||_2^2 < \infty.$$

Hence, $\sum_{n=1}^{\infty} (\frac{f_n}{n})^2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$.

Proof (b): Since $\sum_{n=1}^{\infty} (\frac{f_n}{n})^2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$, then $\sum_{n=1}^{\infty} (\frac{f_n}{n})^2 < \infty$ μ a.e. and as a result $\lim_{n\to\infty} (\frac{f_n}{n}) = 0$ μ a.e.

(2) Consider the measure space $((0,\infty), \mathcal{B}((0,\infty)), \lambda)$, where $\mathcal{B}((0,\infty))$ is the restriction of the Borel σ-algebra, and λ Lebesgue measure restricted to $(0,\infty)$. Determine the value of

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\cos(x^5)}{1+nx^2} \, d\lambda(x).$$

Proof: Let $u_n(x) = \mathbf{1}_{(0,n)} \frac{\cos(x^5)}{1+nx^2}$ and

$$g(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 1/x^2 & \text{if } x > 1. \end{cases}$$

Then, $\lim_{n\to\infty} u_n(x) = 0$ for all $x > 0$, and $|u_n| \leq g$. Furthermore the function g is measurable, non-negative and the improper Riemann integrable on $(0,\infty)$ exists, it follows that it is Lebesgue integrable on $(0,\infty)$. By Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\cos(x^5)}{1+nx^2} \, d\lambda(x) = \lim_{n\to\infty} \int u_n(x) d\lambda(x) = \int \lim_{n\to\infty} u_n(x) d\lambda(x) = 0.$$

(3) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Suppose $f_n, g_n, f, g \in \mathcal{M}(\mathcal{A})$ $(n \geq 1)$ satisfy the following:

(i) $f_n \xrightarrow{\mu} f$,

(ii) $g_n \xrightarrow{\mu} g$,

(iii) $|f_n| \leq C$ for all n, where $C > 0$.

Prove that $f_n g_n \xrightarrow{\mu} fg$.

Proof: Let $\epsilon > 0$ and $\delta > 0$, since μ is a finite measure, it is enough to show that there exists $N \geq 1$ such that

$$\mu(\{x \in X : |f_n g_n - fg| > \epsilon\}) < \delta, \quad \text{for all } n \geq N.$$

First note that

$$|f_n g_n - fg| \leq |f_n||g_n - g| + |g||f_n - f|,$$

thus,

$$\mu(\{x \in X : |f_n g_n - fg| > \epsilon\}) \leq \mu(\{x \in X : |f_n||g_n - g| > \epsilon/2\}) + \mu(\{x \in X : |g_n||f_n - f| > \epsilon/2\}).$$

Let $E_n = \{x \in X : |g| > n\}$, then $E_1 \supseteq E_2 \supseteq \cdots$, and since g is real valued we have $\bigcap_{n=1}^{\infty} E_n = \emptyset$. By finiteness of μ, we have

$$\lim_{n\to\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n) = 0.$$

Choose m large enough so that $\mu(E_m) < \delta/3$. By properties (i) and (ii), there exists $N \geq 1$ so that for $n \geq N$,

$$\mu(\{x \in X : |f_n - f| > \epsilon/2m\}) < \delta/3, \text{ and } \mu(\{x \in X : |g_n - g| > \epsilon/2C\}) < \delta/3.$$

Then for all $n \geq N$,

$$\mu(\{x \in X : |f_n||g_n - g| > \epsilon/2\}) \leq \mu(\{x \in X : |g_n - g| > \epsilon/2C\}) < \delta/3,$$

and

$$\mu(\{x \in X : |g||f_n - f| > \epsilon/2\}) \leq \mu(E_m) + \mu(\{x \in E_m^c : |f_n - f| > \epsilon/2m\}) < 2\delta/3.$$

Therefore, $\mu(\{x \in X : |f_n g_n - fg| > \epsilon\}) < \delta$ for all $n \geq N$, and hence $f_n g_n \xrightarrow{\mu} fg$.

(4) Let $(X, \mathcal{A})$ be a measurable space and μ, ν are finite measure on $\mathcal{A}$. Show that there exists a function $f \in \mathcal{L}^1_+(\mu) \cap \mathcal{L}^1_+(\nu)$ such that for every $A \in \mathcal{A}$, we have

$$\int_A (1 - f)\, d\mu = \int_A f\, d\nu.$$

Proof: First note that $\mu + \nu$ is a measure (Exercise 4.6(ii)), and that $\mu \ll \mu + \nu$. By using a standard argument (first checking indictor functions, then simple functions, then positive functions, then general integrable functions) one sees that for any $g \in \mathcal{L}^1(\mu + \nu)$ one has $g \in \mathcal{L}^1(\mu) \cap \mathcal{L}^1(\nu)$, and

$$\int g\, d(\mu + \nu) = \int g\, d\mu + \int g\, d\nu.$$

Now the condition $\int_A (1-f)\, d\mu = \int_A f\, d\nu$ is equivalent to $\mu(A) = \int_A f\, d(\mu+\nu)$. Since $\mu \ll \mu+\nu$, then by Radon-Nikodym Theorem there exists $f \in \mathcal{L}^1_+(\mu + \nu)$ such that $\mu(A) = \int_A f\, d(\mu + \nu)$. Thus, $f \in \mathcal{L}^1_+(\mu) \cap \mathcal{L}^1_+(\nu)$ and $\int_A (1 - f)\, d\mu = \int_A f\, d\nu$ for all $A \in \mathcal{A}$.

(5) Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x, t) = e^{-xt}$) that $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$, where λ denotes Lebesgue measure.

Proof Let $f : [a, b] \times [0, \infty)$ be given by $f(x, t) = e^{-xt}$. Then f is continuous (hence measurable) and $f > 0$. By Toneli's theorem

$$\int_{[0,\infty)} \int_{[a,b]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{[a,b]} \int_{[0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x).$$

For each fixed $x \in [a, b]$, the function $t \to e^{-xt}$ is positive measurable and the improper Riemann integrable on $[0, \infty)$ exists, so that

$$\int_{[0,\infty)} e^{-xt} d\lambda(t) = \int_0^\infty e^{-xt} dt = \frac{1}{x}.$$

Furthermore, the function $x \to \dfrac{1}{x}$ is measurable and Riemann integrable on $[a, b]$, thus

$$\int_{[a,b]} \int_{[0,\infty)} e^{-xt} d\lambda(t)\, d\lambda(x) = \int_{[a,b]} \frac{1}{x}\, d\lambda(x) = \int_a^b \frac{1}{x}\, dx = \log(b/a).$$

On the other hand,

$$\int_{[0,\infty)} \int_{[a,b]} e^{-xt} d\lambda(x)\, d\lambda(t) = \int_{[0,\infty)} \int_a^b e^{-xt} dx\, d\lambda(t) = \int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t).$$

Therefore, $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$.

(6) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f_n, f \in \mathcal{M}(\mathcal{A})$, $n \geq 1$. Show that f_n converges to f in μ measure **if and only if** $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$.

Solution: First note that $\frac{|f_n - f|}{1 + |f_n - f|} \leq 1$ for all $n \geq 1$, and since $\mu(X) < \infty$ we have $1 \in \mathcal{L}^1(\mu)$.

Now assume that $f_n \xrightarrow{\mu} f$, and let $\epsilon, \delta > 0$, then there exists N such that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \delta\}) < \epsilon, \quad \text{for all } n \geq N.$$

Let $A = \{x \in X : |f_n(x) - f(x)| > \delta\}$, then for all $n \geq N$

$$\int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = \int_A \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu + \int_{A^c} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \leq \int_A 1\, d\mu + \int_{A^c} \delta\, d\mu.$$

Thus, for all $n \geq N$

$$\int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \leq \epsilon + \delta\mu(X).$$

Thus, $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$.

Conversely, assume $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$, and let $\epsilon > 0$. There exists N such that

$$\int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu < \epsilon^2/(1 + \epsilon), \quad \text{for all } n \geq N.$$

Observe first that

$$|f_n - f| > \epsilon \iff \frac{|f_n - f|}{1 + |f_n - f|} > \frac{\epsilon}{1 + \epsilon}.$$

Thus, by Markov Inequality, we have for all $n \geq N$

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) = \mu(\{x \in X : \frac{|f_n - f|}{1 + |f_n - f|} > \frac{\epsilon}{1 + \epsilon}\}) \leq \frac{1 + \epsilon}{\epsilon} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu < \epsilon.$$

Thus, $f_n \xrightarrow{\mu} f$.