Measure and Integration: Retake Final 2015-16

(1) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $f \in \mathcal{M}(\mathcal{A})$. Show that for every $\epsilon > 0$, there exists a set $A \in \mathcal{A}$ and $k \geq 1$ such that $\mu(A) < \epsilon$ and $|f(x)| \leq k$ for all $x \in A^c$. (1 pt)

Proof: Let $D_n = \{x \in X : |f(x)| \leq n\}$. then $D_n \nearrow X$. By Lower continuity of μ, we have

$$\infty > \mu(X) = \mu\left(\bigcup_{n=1}^{\infty} D_n\right) = \lim_{n\to\infty} \mu(D_n).$$

So given $\epsilon > 0$ there exists k sufficiently large so that

$$\mu(D_k) > \mu(X) - \epsilon.$$

Let $A = D_k^c$, then $\mu(A) = \mu(X) - \mu(D_k) < \epsilon$, and for all $x \in A^c = D_k$ we have $|f(x)| \leq k$.

(2) Consider the measure space $[0,1], \mathcal{B}([0,1]), \lambda)$ where λ is Lebesgue measure on $[0,1]$. Define $u_n(x) = \dfrac{nx}{1+n^2x^2}$ for $x \in [0,1]$ and $n \geq 1$. Show that

$$\lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\, d\lambda(x) = 0.$$

(1.5 pts)

Proof: Note that u_n is continuous, hence measurable. Furtheremore $\lim\limits_{n\to\infty} u_n(x) = 0$ for all $x \in [0,1]$. Now the inequality $(1-nx)^2 \geq 0$ implies that $1 + n^2x^2 \geq 2nx$, and hence $u_n(x) \leq 1/2$ for all n and $x \in [0,1]$. Since λ is a finite measure, the constant function $w(x) = 1/2$ is in $\mathcal{L}^1_+(\lambda)$. This implies $u_n \in \mathcal{L}^1_+(\lambda)$, and by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\, d\lambda(x) = \int_{[0,1]} \lim_{n\to\infty} \frac{nx}{1+n^2x^2}\, d\lambda(x) = 0.$$

(3) Let μ and ν be finite measures on $(X, \mathcal{A})$. Show that μ and ν are mutually singular **if and only if** for every $\epsilon > 0$, there exists a set $E \in \mathcal{A}$ such that $\mu(E) < \epsilon$ and $\nu(E^c) < \epsilon$. (2 pts)

Proof: Suppose μ and ν are mutually singular, then there exists a set $A \in \mathcal{A}$ such that $\mu(A) = \nu(A^c) = 0$. Given any $\epsilon > 0$, take $E = A$. Trivially $\mu(E) = \nu(E^c) < \epsilon$.

Conversely, assume the condition. Then for any $n \geq 1$, there exists a set $A_n \in \mathcal{A}$ such that $\mu(A_n) < 1/2^n$ and $\nu(A_n^c) < 1/2^n$. Let $A = \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=m}^{\infty} A_n$. Since $\sum\limits_{n=1}^{\infty} \mu(A_n) \leq \sum\limits_{n=1}^{\infty} 1/2^n = 1 < \infty$, then by Borel-Cantelli Lemma $\mu(A) = 0$. Similarly, if we define $B = \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=m}^{\infty} A_n^c$, then $\nu(B) = 0$. Observe that $A^c \subseteq B$, hence $\nu(A^c) = 0$ and μ and ν are mutually singular.

(4) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(u_n)_n \subset \mathcal{L}^p(\mu)$ converging in $\mathcal{L}^p(\mu)$ to a function $u \in \mathcal{L}^p(\mu)$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{A}$ with $\mu(A) < \delta$, then $\int_A |u_n|^p\, d\mu < \epsilon$ for all $n \geq 1$. (2 pts)

Proof: Let $\epsilon > 0$. By assupmtion $\lim\limits_{n\to\infty} ||u_n - u||_p = 0$, hence there exists $N \geq 1$ such that

$$||u_n - u||_p = \int |u_n - u|^p\, d\mu < \epsilon/2^{p+1}, \text{ for all } n \geq N.$$

Note that the measures $\nu(A) = \int_A |u|^p \, d\mu$ and $\nu_i(A) = \int_A |u_i|^p \, d\mu$, $i = 1, \cdots, N-1$ are absolutely continuous with respect to μ, hence there exists a $\delta > 0$ such that if $A \in \mathcal{A}$ with $\mu(A) < \delta$ then

$$\int_A |u|^p \, d\mu < \epsilon/2^{p+1}, \text{ and } \int_A |u_i|^p \, d\mu < \epsilon, \; i = 1, \cdots, N-1.$$

Recall that $|u_i|^p \leq 2^p \Big(|u|^p + |u_i - u|^p \Big)$. For $i \geq N$, and $A \in \mathcal{A}$ with $\mu(A) < \delta$ we have,

$$\int_A |u_i|^p \, d\mu \leq 2^p \int_A |u|^p \, d\mu + 2^p \int |u_i - u|^p \, d\mu < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence the result holds for all i.

(5) Consider the measure space $([0,\infty), \mathcal{B}([0,\infty)), \lambda)$, where $\mathcal{B}([0,\infty))$ is the Borel σ-algebra, and λ is Lebesgue measure on $[0,\infty)$. Let $f(x,y) = ye^{-(1+x^2)y^2}$ for $0 \leq x, y < \infty$.

(a) Show that $f \in \mathcal{L}^1(\lambda \times \lambda)$, and determine the value of $\int_{[0,\infty)\times[0,\infty)} f \, d(\lambda \times \lambda)$. (1 pt)

(b) Prove that $\int_{[0,\infty)\times[0,\infty)} f \, d(\lambda \times \lambda) = \Big(\int_{[0,\infty)} e^{-x^2} \, d\lambda(x) \Big)^2$. Use part (a) to deduce the value of $\int_{[0,\infty)} e^{-x^2} \, d\lambda(x)$. (1 pt)

Proof(a): The function f is non-negative and continuous, and hence measurable. For each fixed $x \geq 0$, the improper Riemann-integral of the function $y \to ye^{-(1+x^2)y^2}$ exists, hence

$$\int_{[0,\infty)} ye^{-(1+x^2)y^2} \, d\lambda(y) = (R)\int_0^\infty ye^{-(1+x^2)y^2} \, dy = \frac{1}{2(1+x^2)}.$$

Similarly the improper Riemann-integral of the function $x \to \dfrac{1}{2(1+x^2)}$ exists and hence

$$\int_{[0,\infty)} \frac{1}{2(1+x^2)} \, d\lambda(x) = (R)\int_0^\infty \frac{1}{2(1+x^2)} \, dx = \frac{\pi}{4}.$$

By Tonelli's Theorem

$$\int_{[0,\infty)\times[0,\infty)} f \, d(\lambda \times \lambda) = \int_{[0,\infty)} \int_{[0,\infty)} ye^{-(1+x^2)y^2} \, d\lambda(y) \, d\lambda(x) = \frac{\pi}{4}.$$

This implies that $f \in \mathcal{L}^1(\lambda \times \lambda)$, and the integral has value $\frac{\pi}{4}$.

Proof(b): First note that with a simple substitution $u = xy$, one has by Theorem 7.10 that

$$\int_{[0,\infty)} ye^{-(1+x^2)y^2} \, d\lambda(x) = \int_{[0,\infty)} e^{-y^2} e^{-u^2} \, d\lambda(u).$$

Hence,

$$\int_{[0,\infty)} \int_{[0,\infty)} ye^{-(1+x^2)y^2} \, d\lambda(x) \, d\lambda(y) = \Big(\int_{[0,\infty)} e^{-y^2} \, d\lambda(y) \Big)^2.$$

By Tonelli's Theorem we have

$$\int_{[0,\infty)\times[0,\infty)} f \, d(\lambda \times \lambda) = \int_{[0,\infty)} \int_{[0,\infty)} ye^{-(1+x^2)y^2} \, d\lambda(x) \, d\lambda(y) = \Big(\int_{[0,\infty)} e^{-y^2} \, d\lambda(y) \Big)^2.$$

From part (a), we have

$$\Big(\int_{[0,\infty)} e^{-y^2} \, d\lambda(y) \Big)^2 = \frac{\pi}{4},$$

hence

$$\int_{[0,\infty)} e^{-y^2} \, d\lambda(y) = \sqrt{\frac{\pi}{4}}.$$

(6) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and Let $(u_j)_j \subseteq \mathcal{L}^p(\mu)$, $p \geq 1$. Suppose $(u_j)_j$ converges to u μ a.e., and that the sequence $((u_j^p)^-)$ is uniformly integrable. Prove that $\liminf\limits_{n\to\infty} \int u_n^p \, d\mu \geq \int u^p \, d\mu$. (1.5 pts)

Proof Since (u_j) converges to u μ a.e., then (u_j^p) converges to u^p μ a.e and $((u_j^p)^-)_j$ converges μ a.e. and hence in μ measure to $(u^p)^-$. By Vitali's Theorem applied to the sequence $(u_j^-)_j$, we have that $(u_j^-)_j$ converges to u^- in $\mathcal{L}^p(\mu)$, and hence $\lim\limits_{j\to\infty} \int (u_j^p)^- \, d\mu = \int (u^p)^- \, d\mu$, and $u \in \mathcal{L}^p(\mu)$. From the above we have

$$\liminf_{j\to\infty} \int u_j^p \, d\mu = \liminf_{j\to\infty} \int (u_j^p)^+ \, d\mu - \lim_{j\to\infty} \int (u_j^p)^- \, d\mu = \liminf_{j\to\infty} \int (u^p)_j^+ \, d\mu - \int (u^p)^- \, d\mu.$$

Since $((u_j^p)^+)$ converges to $(u^p)^+$ μ a.e., by Fatous Lemma

$$\liminf_{j\to\infty} \int (u_j^p)^+ \, d\mu \geq \int \liminf_{j\to\infty} (u_j^p)^+.$$

From the above we have

$$\liminf_{j\to\infty} \int u_j^p \, d\mu \geq \int (u_j^p)^+ \, d\mu - \int (u_j^p)^- \, d\mu = \int u^p \, d\mu.$$