Measure and Integration: Solutions Hertentamen 2014-15

(1) Consider the measure space $([0,1),\mathcal{B}([0,1)),\lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define the transformation $T:[0,1)\to[0,1)$ given by

$$T(x) = \begin{cases} 3x & 0\le x<1/3, \\ 3x-1, & 1/3\le x<2/3 \\ 3x-2, & 2/3\le x<1. \end{cases}$$

(a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable. (0.5 pts)
(b) Determine the image measure $T(\lambda)=\lambda\circ T^{-1}$. (0.5 pts)
(c) Show that for all $f\in\mathcal{L}^1(\lambda)$ one has, $\int f\,d\lambda=\int f\circ T\,d\lambda$. (0.5 pts)
(d) Let $\mathcal{C}=\{A\in\mathcal{B}([0,1)):\lambda(T^{-1}A\Delta A)=0\}$. Show that $\mathcal{C}$ is a σ-algebra. (0.5 pts)

Solution(a): To show T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable, it is enough to consider inverse images of intervals of the form $[a,b)\subset[0,1)$. Now,

$$T^{-1}([a,b))=[\frac{a}{3},\frac{b}{3})\cup[\frac{a+1}{3},\frac{b+1}{3})\cup[\frac{a+2}{3},\frac{b+2}{3})\in\mathcal{B}([0,1)).$$

Thus, T is measurable.

Solution(b): We claim that $T(\lambda)=\lambda$. To prove this, we use Theorem 5.7. Notice that $\mathcal{B}([0,1))$ is generated by the collection $\mathcal{G}=\{[a,b):0\le a\le b<1\}$ which is closed under finite intersections. Now,

$$\begin{aligned} T(\lambda)([a,b)) &= \lambda(T^{-1}([a,b))) \\ &= \lambda([\frac{a}{3},\frac{b}{3}))+\lambda([\frac{a+1}{3},\frac{b+1}{3}))+\lambda([\frac{a+2}{3},\frac{b+2}{3})) \\ &= b-a=\lambda([a,b)). \end{aligned}$$

Since the constant sequence $([0,1))$ is exhausting, belongs to $\mathcal{G}$ and $\lambda([0,1))=T(\lambda([0,1))=1<\infty$, we have by Theorem 5.7 that $T(\lambda)=\lambda$.

Solution(c): We use a standard argument. Assume first that $f=\mathbf{1}_A$ for some $A\in\mathcal{B}([0,1)$. Note that $\mathbf{1}_{T^{-1}A}=\mathbf{1}_A\circ T$, hence by part (b),

$$\int f\,d\lambda=\int\mathbf{1}_A\,d\lambda=\lambda(A)=\lambda(T^{-1}A)=\int\mathbf{1}_{T^{-1}A}\,d\lambda=\int\mathbf{1}_A\circ T\,d\lambda=\int f\circ T\,d\lambda.$$

Now, let $f\in\mathcal{E}^+$, and let $f=\sum_{i=1}^n a_i\mathbf{1}_{A_i}$ be a standard representation of f. By linearity of the integral and the above, we have

$$\int f\,d\lambda=\sum_{i=1}^n a_i\int\mathbf{1}_{A_i}\,d\lambda=\sum_{i=1}^n a_i\int\mathbf{1}_{A_i}\circ T\,d\lambda=\int\sum_{i=1}^n a_i\mathbf{1}_{A_i}\circ T\,d\lambda=\int f\circ T\,d\lambda.$$

Assume now that $f\in\mathcal{L}^1_+(\lambda)$, then there exists an increasing sequence $(f_n)_n$ in $\mathcal{E}^+$ such that $f=\sup_n f_n$. By Beppo-Levi, we have

$$\int f\,d\lambda=\sup_n\int f_n\,d\lambda=\sup_n\int f_n\circ T\,d\lambda=\int\sup_n f_n\circ T\,d\lambda=\int f\circ T\,d\lambda.$$

Finally, consider $f\in\mathcal{L}^1(\lambda)$, then $f^+,f^-\in\mathcal{L}^1_+(\lambda)$, and

$$\int f\,d\lambda=\int f^+\,d\lambda-\int f^-\,d\lambda=\int f^+\circ T\,d\lambda-\int f^-\circ T\,d\lambda=\int f\circ T\,d\lambda.$$

Solution(d): We check the three conditions for a collection of sets to be a σ-algebra. Firstly, $[0,1) \in \mathcal{B}([0,1))$ and $T^{-1}([0,1)) = [0,1)$. Thus $\lambda(T^{-1}([0,1))\Delta[0,1)) = \lambda(\emptyset) = 0$ so that $[0,1) \in \mathcal{C}$. Secondly, Let $A \in \mathcal{C}$, then $\lambda(T^{-1}A\Delta A) = 0$. Since T is a measurable function and $A \in \mathcal{B}([0,1))$, then $(T^{-1}A)^c) \in \mathcal{B}([0,1))$. Since $(T^{-1}A)^c) = T^{-1}A^c$, and $T^{-1}A\Delta A = T^{-1}A^c\Delta A^c$, we have $\lambda(T^{-1}A^c\Delta A^c) = \lambda(T^{-1}A\Delta A) = 0$, so $A^c \in \mathcal{C}$. Thirdly, let (A_n) be a sequence in $\mathcal{C}$, then $A_n \in \mathcal{B}([0,1))$ and $\lambda(T^{-1}A_n\Delta A_n) = 0$ for each n. Since $\mathcal{B}([0,1))$ is a σ-algebra, we have $\bigcup_n A_n \in \mathcal{B}([0,1))$. Note that

$$T^{-1}(\bigcup_n A_n) = \bigcup_n T^{-1}A_n \text{ and } T^{-1}(\bigcup_n A_n)\Delta\bigcup_n A_n \subseteq \bigcup_n (T^{-1}A_n\Delta A_n).$$

Thus,

$$\lambda\Big(T^{-1}(\bigcup_n A_n)\Delta\bigcup_n A_n\Big) \leq \sum_n \lambda\Big(T^{-1}A_n\Delta A_n\Big) = 0.$$

Hence, $\bigcup_n A_n \in \mathcal{C}$. This shows that $\mathcal{C}$ is a σ-algebra.

(2) Consider the measure space $((0,\infty), \mathcal{B}((0,\infty)), \lambda)$, where $\mathcal{B}((0,\infty))$ is the restriction of the Borel σ-algebra, and λ Lebesgue measure restricted to $(0,\infty)$. Determine the value of

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\cos(x^5)}{1+nx^2}\, d\lambda(x).$$

(2 pts)

Proof: Let $u_n(x) = \mathbf{1}_{(0,n)} \frac{\cos(x^5)}{1+nx^2}$ and

$$g(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 1/x^2 & \text{if } x > 1. \end{cases}$$

Then, $\lim_{n\to\infty} u_n(x) = 0$ for all $x > 0$, and $|u_n| \leq g$. Furthermore the function g is measurable, non-negative and the improper Riemann integrable on $(0,\infty)$ exists, it follows that it is Lebesgue integrable on $(0,\infty)$. By Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\cos(x^5)}{1+nx^2}\, d\lambda(x) = \lim_{n\to\infty} \int u_n(x) d\lambda(x) = \int \lim_{n\to\infty} u_n(x) d\lambda(x) = 0.$$

(3) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $1 < p, q < \infty$ two conjugate numbers (i.e. $1/p + 1/q = 1$). Let $g \in \mathcal{M}(\mathcal{A})$ be a measurable function satisfying

$$\int |fg|\, d\mu \leq C||f||_p$$

for all $f \in \mathcal{L}^p(\mu)$ and for some constant C.

(a) For $n \geq 1$, let $E_n = \{x \in X : |g(x)| \leq n\}$ and $g_n = \mathbf{1}_{E_n}|g|^{q/p}$. Show that $g_n \in \mathcal{L}^p(\mu)$ for all $n \geq 1$. (0.5 pts)

(b) Show that $g \in \mathcal{L}^q(\mu)$. (1.5 pts)

Proof(a):

$$\int |g_n|^p\, d\mu = \int \mathbf{1}_{E_n}|g|^q\, d\mu \leq n^q\mu(E_n) < \infty\ n \geq 1.$$

Thus, $g_n \in \mathcal{L}^p(\mu)$ for all $n \geq 1$.

Proof(b): Since $g_n \in \mathcal{L}^p(\mu)$ for all $n \geq 1$, then by hypothesis,

$$\int |g_n g|\, d\mu \leq C||g_n||_p.$$

However, $|g_n g| = \mathbf{1}_{E_n}|g|^q = |g_n|^p$, and $||g_n||_p^p = ||\mathbf{1}_{E_n}g||_q^q$. Substituting these in the above inequality, we get

$$||\mathbf{1}_{E_n}g||_q^q = \int \mathbf{1}_{E_n}|g|^q\, d\mu \leq C||\mathbf{1}_{E_n}g||_q^{q/p},$$

implying $||\mathbf{1}_{E_n}g||_q \leq C$. Since (E_n) is an increasing sequence of measurable sets with $\bigcup_{n=1}^\infty E_n = X$, then $\mathbf{1}_{E_n}|g|^q \nearrow |g|^q$. By Beppo-Levi we have $||g||_q \leq C$, and hence $g \in \mathcal{L}^q(\mu)$.

(4) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and (f_j) a uniformly integrable sequence of measurable functions. Define $F_k = \sup_{1\leq j\leq k} |f_j|$ for $k \geq 1$.

(a) Show that for any $w \in \mathcal{M}^+(\mathcal{A})$,

$$\int_{\{F_k>w\}} F_k \, d\mu \leq \sum_{j=1}^{k} \int_{\{|f_j|>w\}} |f_j| \, d\mu.$$

(0.5 pts)

(b) Show that for every $\epsilon > 0$, there exists a $w_\epsilon \in \mathcal{L}^1_+(\mu)$ such that for all $k \geq 1$

$$\int_X F_k \, d\mu \leq \int_X w_\epsilon \, d\mu + k\epsilon.$$

(1 pt)

(c) Show that

$$\lim_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = 0.$$

(0.5 pts)

Proof (a) Let $w \in \mathcal{M}^+(\mathcal{A})$, then

$$\begin{aligned} \int_{\{F_k>w\}} F_k \, d\mu &\leq \sum_{j=1}^{k} \int_{\{F_k>w\}\cap\{|f_j|=F_k\}} F_k \, d\mu \\ &\leq \sum_{j=1}^{k} \int_{\{|f_j|>w\}} |f_j.| \, d\mu. \end{aligned}$$

Proof (b) Let $\epsilon > 0$. By uniform integrability of the sequence (f_j) there exists $w_\epsilon \in \mathcal{L}^+(\mu)$ such that

$$\int_{\{|f_j|>w_\epsilon\}} |f_j| \, d\mu < \epsilon$$

for all $j \geq 1$. By part (a)

$$\int_{\{F_k>w_\epsilon\}} F_k \, d\mu \leq \sum_{j=1}^{k} \int_{\{|f_j|>w_\epsilon\}} |f_j| \, d\mu \leq k\epsilon.$$

Now,

$$\begin{aligned} \int_X F_k \, d\mu &= \int_{\{F_k>w_\epsilon\}} F_k \, d\mu + \int_{\{F_k\leq w_\epsilon\}} F_k \, d\mu \\ &\leq k\epsilon + \int_X w_\epsilon \, d\mu. \end{aligned}$$

Proof (c) For any $\epsilon > 0$, by part (b),

$$\frac{1}{k} \int_X F_k \, d\mu \leq \frac{1}{k} \int_X w_\epsilon \, d\mu + \epsilon.$$

Thus,

$$\limsup_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu \leq \epsilon,$$

for any ϵ. Since $F_k \geq 0$, we see that

$$\limsup_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = \lim_{k\to\infty} \frac{1}{k} \int_X F_k \, d\mu = 0.$$

(5) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Let $k, g \in \mathcal{L}^1(\lambda)$ and define $F : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R} \to \overline{\mathbb{R}}$ by

$$F(x, y) = k(x - y)g(y).$$

(a) Show that F is measurable. (1 pt)

(b) Show that $F \in \mathcal{L}^1(\lambda \times \lambda)$, and

$$\int_{\mathbb{R}\times\mathbb{R}} F(x,y)d(\lambda\times\lambda)(x,y) = \left(\int_{\mathbb{R}} k(x)d\lambda(x)\right)\left(\int_{\mathbb{R}} g(y)d\lambda(y)\right).$$

(1 pts)

Proof(a): To show measurablity of F, we first extend the domain of g to $\mathbb{R}^2$ as follows. Define $\overline{g} : \mathbb{R}^2 \to \mathbb{R}$ by $\overline{g}(x,y) = g \circ \pi_2(x,y) = g(y)$. It is easy to see that $\overline{g}$ is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable. Moreover, the function $d : \mathbb{R}^2 \to \mathbb{R}$ given by $d(x,y) = x - y$ is continuous hence $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable. Since

$$F(x,y) = k(x-y)g(y) = k \circ d(x,y)\overline{g}(x,y)$$

is the product of two $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable functions, it follows that F is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable.

Proof(b): Since Lebesgue measure is translation invariant, we have

$$\begin{aligned}\int\int |F(x,y)|\, d\lambda(x)d\lambda)(y) &= \int\int |k(x-y)||g(y)|\, d\lambda(x)d\lambda(y) \\ &= \int\int |k(x)||g(y)|\, d\lambda(x)d\lambda)(y) \\ &= \int |k(x)|\, d\lambda(x) \int |g(y)|d\lambda)(y) < \infty.\end{aligned}$$

By Fubini's Theorem, this implies that F is $\lambda \times \lambda$ integrable, and

$$\begin{aligned}\int F(x,y)\, d(\lambda\times\lambda)(x,y) &= \int\int k(x-y)g(y)\, d\lambda(x)\, d\lambda(y) \\ &= \int\int k(x)g(y)\, d\lambda(x)d\lambda)(y) \\ &= \int k(x)\, d\lambda(x) \int g(y)d\lambda)(y).\end{aligned}$$