Measure and Integration: Solutions Quiz 2014-15

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{3k + 2^n}{2^n} \cdot \mathbf{1}_{[k/2^n,(k+1)/2^n)}(x),\ n \geq 1.$$

   (a) Show that f_n is measurable, and $f_n(x) \leq f_{n+1}(x)$ for all $x \in \mathbb{R}$. (1 pt)

   (b) Show that $\int \sup_{n\geq 1} f_n \, d\lambda = \frac{5}{2}$. (2 pts)

Solution(a): Since $[k/2^n, (k+1)/2^n) \in \mathcal{B}(\mathbb{R})$, then $1_{[k/2^n,(k+1)/2^n)}$ is a measurable function. Thus f_n is a linear combination of measurable functions (in fact f_n is a simple function) and hence measurable. For $x \notin [0, 1)$, we have $f_n(x) = f_{n+1}(x) = 0$. Suppose $x \in [0, 1)$, then there exists a $0 \leq k \leq 2^n - 1$ such that $x \in [k/2^n, (k+1)/2^n)$. Since

$$[k/2^n, (k + 1)/2^n) = [2k/2^{n+1}, (2k + 1)/2^{n+1}) \cup [(2k + 1)/2^{n+1}, (2k + 2)/2^{n+1}),$$

we see that $f_n(x) = \frac{3k}{2^n} + 1$ while $f_{n+1}(x) \in \{\frac{3(2k)}{2^{n+1}} + 1, \frac{3(2k+1)}{2^{n+1}} + 1\}$ so that $f_n(x) \leq f_{n+1}(x)$.

Solution(b): We apply Beppo-Levi,

$$\begin{aligned}
\int \sup_{n\geq 1} f_n \, d\lambda &= \sup_{n\geq 1} \int f_n \, d\lambda \\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\frac{3k}{2^n} + 1)\lambda([k/2^n, (k + 1)/2^n)) \\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} (\frac{3k}{2^n} + 1)\frac{1}{2^n} \\
&= \lim_{n\to\infty} \frac{3}{4^n} \sum_{k=0}^{2^n-1} k + 1 \\
&= \lim_{n\to\infty} \frac{3}{2}\frac{(2^n - 1)2^n}{4^n} + 1 = \frac{5}{2}.
\end{aligned}$$

2. Let X be a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$. Consider $\sigma(\mathcal{C})$, the smallest σ-algebra over X containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(\mathcal{C}_0)$.

   (a) Show that $\mathcal{D}$ is a σ-algebra over X.

(b) Show that $\mathcal{D} = \sigma(\mathcal{C})$.

Proof (a): Clearly $\emptyset \in \mathcal{D}$ since $\emptyset$ belongs to every σ-algebra. Let $A \in \mathcal{D}$, then there is a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $A \in \sigma(\mathcal{C}_0)$. But then $A^c \in \sigma(\mathcal{C}_0)$, hence $A^c \in \mathcal{D}$. Finally, let $\{A_n\}$ be in $\mathcal{D}$, then for each n there exists a countable collection $\mathcal{C}_n \subseteq \mathcal{C}$ such that $A_n \in \sigma(\mathcal{C}_n)$. Let $\mathcal{C}_0 = \bigcup_n \mathcal{C}_n$, then $\mathcal{C}_0 \subseteq \mathcal{C}$, and $\mathcal{C}_0$ is countable. Furthermore, $\sigma(\mathcal{C}_n) \subseteq \sigma(\mathcal{C}_0)$, and hence $A_n \in \sigma(\mathcal{C}_0)$ for each n which implies that $\bigcup_n A_n \in \sigma(\mathcal{C}_0)$. Therefore, $\bigcup_n A_n \in \mathcal{D}$ and $\mathcal{D}$ is a σ-algebra.

Proof (b): By definition $\mathcal{D} \subseteq \sigma(\mathcal{C})$. Also, $\mathcal{C} \subseteq \mathcal{D}$ since $C \in \sigma(\{C\})$ for every $C \in \mathcal{C}$. Since $\sigma(\mathcal{C})$ is the smallest σ-algebra over X containg $\mathcal{C}$, then by part (a) $\sigma(\mathcal{C}) \subseteq \mathcal{D}$. Thus, $\mathcal{D} = \sigma(\mathcal{C})$.

3. Let $(X, \mathcal{A}, \mu)$ be a **finite** measure space with $0 < \mu(X) < \infty$), and $T : X \to X$ an $\mathcal{A}/\mathcal{A}$-measurable function satisfying $\mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{A}$. For $n \geq 1$, denote by $T^n = T \circ T \circ \cdots \circ T$ the n-fold composition of T with itself.

   (a) For $B \in \mathcal{A}$, let $D(B) = \{x \in B : T^n(x) \notin B \text{ for all } n \geq 1\}$. Show that $D(B) \in \mathcal{A}$.

   (b) For $n \geq 1$, let $D(B)_n = T^{-n}(D(B))$. Show that $\mu(D(B)_n) = \mu(D(B))$, for $n \geq 1$, and that $D(B)_n \cap D(B)_m = \emptyset$ if $n \neq m$.

   (c) Show that $\mu(D(B)) = 0$.

   (d) Suppose $A \in \mathcal{A}$ satisfies the property that if $B \in \mathcal{A}$ with $\mu(B) > 0$, then there exists $n \geq 1$ such that $\mu(A \cap T^{-n}B) > 0$. Show that $\mu(A) > 0$, and if additionally $T^{-1}(A) = A$, then $\mu(A) = \mu(X)$.

Proof (a):Since the composition of measurable functions is measurable, we see that $T^{-n}(B^c) \in \mathcal{A}$ for all $n \geq 1$. Thus, $D(B) = B \cap \bigcup_{n=1}^{\infty} T^{-n}(B^c) \in \mathcal{A}$.

Proof (b): The first statement is easily proved by induction. The result is true for $n = 1$, and assume it is true for n, i.e $\mu(D(B)_n) = \mu(D(B))$. Now $D(B)_{n+1} = T^{-1}(D_n)$, so

$$\mu(D(B)_{n+1}) = \mu(T^{-1}(D(B)_n)) = \mu(D(B)_n) = \mu(D(B)).$$

Assume $m < n$. If $x \in D(B)_m \cap D(B)_n$, then from $x \in D(B)_m$ one gets $T^m(x) \in D(B)$, so $T^m(x) \in B$ and $T^n(x) = T^{n-m}(T^m x) \notin B$, contradicting the fact that $x \in D(B)_m$. So $D(B)_n \cap D(B)_m = \emptyset$

Proof (c): Assume $\mu(D(B)) > 0$, then $\mu(D(B)_n) = \mu(D(B)) > 0$. Since the sets $(D(B)_n)$ are mutually disjoint, we have

$$\infty > \mu(X) \geq \mu(\bigcup_{n=1}^{\infty} D(B)_n) = \sum_{n=1}^{\infty} \mu(D(B)_n) = \sum_{n=1}^{\infty} \mu(D(B)) = \infty,$$

leading to a contradiction. Hence, $\mu(D(B)) = 0$.

Proof (d): Since $\mu(X) > 0$, then there exists $n \geq 1$ such that $\mu(A \cap T^{-n}(X)) > 0$. Since $T^{-n}(X) = X$, we have $\mu(A) = \mu(A \cap T^{-n}X) > 0$. Suppose now that $A = T^{-1}(A)$ and assume for the sake of getting a contradiction that $0 < \mu(A) < \mu(X)$, then $0 < \mu(A^c) < \mu(X)$ and also $T^{-1}(A^c) = A^c$. By hypothesis there exists $n \geq 1$ such that $\mu(A \cap A^c) = \mu(A \cap T^{-n}(A^c)) > 0$, leading to a contradiction since $\mu(A \cap A^c) = 0$. Thus, $\mu(A) = \mu(X)$.