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(1) Let $(X, \mathcal{B}, \nu)$ be a measure space, and suppose $X = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(B) = \sum_{n=1}^{\infty} 2^{-n}\nu(B \cap E_n)/(\nu(E_n)+1)$.

(a) Prove that μ is a finite measure on $(X, \mathcal{B})$. (10 pt.)

(b) Let $B \in \mathcal{B}$. Prove that $\mu(B) = 0$ **if and only if** $\nu(B) = 0$. (10 pt.)

Proof (a): Clearly $\mu(\emptyset) = 0$, and

$$\mu(X) = \sum_{n=1}^{\infty} 2^{-n}\nu(E_n)/(\nu(E_n)+1) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.$$

Now, let (C_n) be a disjoint sequence in $\mathcal{B}$. Then,

$$\begin{aligned}
\mu(\textstyle\bigcup_{m=1}^{\infty} C_m) &= \sum_{n=1}^{\infty} 2^{-n}\nu((\bigcup_{m=1}^{\infty} C_m) \cap E_n)/(\nu(E_n)+1) \\
&= \sum_{n=1}^{\infty} 2^{-n} \sum_{m=1}^{\infty} \nu(C_m \cap E_n)/(\nu(E_n)+1) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-n}\nu(C_m \cap E_n)/(\nu(E_n)+1) \\
&= \sum_{m=1}^{\infty} \mu(C_m).
\end{aligned}$$

Thus, μ is a finite measure.

Proof (b): Suppose that $\nu(B) = 0$, then $\nu(B \cap E_n) = 0$ for all n, hence $\mu(B) = 0$. Conversely, suppose $\mu(B) = 0$, then $\nu(B \cap E_n) = 0$ for all n. Since $X = \bigcup_{n=1}^{\infty} E_n$ (disjoint union), then

$$\nu(B) = \nu(B \cap \bigcup_{n=1}^{\infty} E_n) = \nu(\bigcup_{n=1}^{\infty}(B \cap E_n)) = \sum_{n=1}^{\infty} \nu(B \cap E_n) = 0.$$

(2) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Determine the value of $\lim_{n\to\infty} \int_{(0,n)} x^2 \left(1 - \frac{x}{n}\right)^n d\lambda(x)$. (20 pt.)

Proof: Let $u_n(x) = \mathbf{1}_{(0,n)} x^2 \left(1 - \frac{x}{n}\right)^n$, then $\lim_{n\to\infty} u_n(x) = \mathbf{1}_{(0,\infty)} x^2 e^{-x}$. Using the fact that $\left(1 - \frac{x}{n}\right)^n \nearrow e^{-x}$, we see that $u_n(x) \leq \mathbf{1}_{(0,\infty)} x^2 e^{-x}$. Since the function $x^2 e^{-x}$ is measurable, non-negative and the improper Riemann integrable on $[0, \infty)$ exists, it follows that it is Lebesgue integrable on $[0, \infty)$ (and hence also on $(0, \infty)$) and its value equals the improper Riemann integral. By Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{(0,n)} x^2 \left(1 - \frac{x}{n}\right)^n d\lambda(x) &= \lim_{n\to\infty} \int u_n(x) d\lambda(x) \\
&= \int \mathbf{1}_{(0,\infty)} x^2 e^{-x} d\lambda(x) = \int_0^{\infty} x^2 e^{-x}\, dx = 2.
\end{aligned}$$

(3) Let X be a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$. Consider $\sigma(\mathcal{C})$, the smallest σ-algebra over X containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(\mathcal{C}_0)$.

(a) Show that $\mathcal{D}$ is a σ-algebra over X. (12 pt.)

(b) Show that $\mathcal{D} = \sigma(\mathcal{C})$. (8 pt.)

Proof (a): Clearly $\emptyset \in \mathcal{D}$ since $\emptyset$ belongs to every σ-algebra. Let $A \in \mathcal{D}$, then there is a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $A \in \sigma(\mathcal{C}_0)$. But then $A^c \in \sigma(\mathcal{C}_0)$, hence $A^c \in \mathcal{D}$. Finally, let $\{A_n\}$ be in $\mathcal{D}$, then for each n there exists a countable collection $\mathcal{C}_n \subseteq \mathcal{C}$ such that $A_n \in \sigma(\mathcal{C}_n)$. Let $\mathcal{C}_0 = \bigcup_n \mathcal{C}_n$, then $\mathcal{C}_0 \subseteq \mathcal{C}$, and $\mathcal{C}_0$ is countable. Furthermore, $\sigma(\mathcal{C}_n) \subseteq \sigma(\mathcal{C}_0)$, and hence $A_n \in \sigma(\mathcal{C}_0)$ for each n which implies that $\bigcup_n A_n \in \sigma(\mathcal{C}_0)$. Therefore, $\bigcup_n A_n \in \mathcal{D}$ and $\mathcal{D}$ is a σ-algebra.

Proof (b): By definition $\mathcal{D} \subseteq \sigma(\mathcal{C})$. Also, $\mathcal{C} \subseteq \mathcal{D}$ since $C \in \sigma(\{C\})$ for every $C \in \mathcal{C}$. Since $\sigma(\mathcal{C})$ is the smallest σ-algebra over X containg $\mathcal{C}$, then by part (a) $\sigma(\mathcal{C}) \subseteq \mathcal{D}$. Thus, $\mathcal{D} = \sigma(\mathcal{C})$.

(4) Let $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \nu_1)$ be σ-finite measure spaces. Suppose $f \in \mathcal{L}^1(\mu_1)$ and $g \in \mathcal{L}^1(\nu_1)$ are non-negative. Define measures μ_2 on $\mathcal{A}$ and ν_2 on $\mathcal{B}$ by

$$\mu_2(A) = \int_A f\, d\mu_1 \text{ and } \nu_2(B) = \int_B g\, d\nu_1,$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(a) For $D \in \mathcal{A} \otimes \mathcal{B}$ and $y \in Y$, let $D_y = \{x \in X : (x, y) \in D\}$. Show that if $\mu_1(D_y) = 0$ ν_1 a.e., then $\mu_2(D_y) = 0$ ν_2 a.e. (7 pt.)

(b) Show that if $D \in \mathcal{A} \otimes \mathcal{B}$ is such that $(\mu_1 \times \nu_1)(D) = 0$ then $(\mu_2 \times \nu_2)(D) = 0$. (6 pt.)

(c) Show that for every $D \in \mathcal{A} \otimes \mathcal{B}$ one has

$$(\mu_2 \times \nu_2)(D) = \int_D f(x)g(y)\, d(\mu_1 \times \nu_1)(x, y).$$

(7 pt.)

Proof(a) Suppose $\mu_1(D_y) = 0$ ν_1 a.e. Let $B = \{y \in Y : \mu_1(D_y) > 0\}$, and $C = \{y \in Y : \mu_2(D_y) > 0\}$. By our assumption, $\nu_1(B) = 0$. By Theorem 10.9(ii), for any $y \in Y \setminus B$ one has $\mu_2(D_y) = 0$. Thus, $C \subset B$, so that $\nu_1(C) = 0$. Applying Theorem 10.9(ii) again, we see that $\nu_2(C) = 0$. Thus, $\mu_2(D_y) = 0$ ν_2 a.e.

Proof(b) Suppose that $D \in \mathcal{A} \otimes \mathcal{B}$ is such that $(\mu_1 \times \nu_1)(D) = 0$. Then,

$$\int \mu_1(D_y)\, d\nu_1(y) = (\mu_1 \times \nu_1)(D) = 0.$$

By Theorem 10.9(i), we have that $\mu_1(D_y) = 0$ ν_1 a.e. By part (a) above this implies that $\mu_2(D_y) = 0$ ν_2 a.e. Thus, by Theorem 10.9(i)

$$(\mu_2 \times \nu_2)(D) = \int \mu_2(D_y)\, d\nu_2(y) = 0.$$

Proof(c) By Tonelli's Theorem, we have

$$\begin{aligned}
(\mu_2 \times \nu_2)(D) &= \int_Y \int_X \mathbf{1}_{D_y}(x)\, d\mu_2(x)\, d\nu_2(y) \\
&= \int_Y \left(\int_X \mathbf{1}_{D_y}(x) f(x)\, d\mu_1(x) \right) d\nu_2(y) \\
&= \int_Y \left(\int_X \mathbf{1}_{D_y}(x) f(x)\, d\mu_1(x) \right) g(y)\, d\nu_1(y) \\
&= \int_Y \int_X \mathbf{1}_D(x, y) f(x) g(y)\, d\mu_1(x)\, d\nu_1(y) \\
&= \int_{X \times Y} \mathbf{1}_D(x, y) f(x) g(y)\, d(\mu_1 \times \nu_1)(x, y) \\
&= \int_D f(x) g(y)\, d(\mu_1 \times \nu_1)(x, y).
\end{aligned}$$

(5) Let $(X, \mathcal{A}, \mu)$ be a probability space and let $f \in \mathcal{M}(\mathcal{A})$. Suppose $(f_n) \subset \mathcal{M}(\mathcal{A})$ converges in measure to f, i.e. $f_n \xrightarrow{\mu} f$.

(a) Show that there exists a sequence $n_1 < n_2 < \cdots$ such that

$$\mu(\{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}) \leq 2^{-k},$$

for all $k \geq 1$. (8 pt.)

(b) Let $A_k = \{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}$ and $A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Show that $\mu(A) = 0$, and $\lim_{n\to\infty} f_{n_k}(x) = f(x)$ for all $x \notin A$. Conclude that $f_{n_k} \to f$ μ a.e. (12 pt.)

Proof(a) Using convergence in measure, the sequence n_k is defined inductively as follows. Starting with $\epsilon_1 = 1$, we find n_1 such that $\mu(\{x \in X : |f_{n_k}(x) - f(x)| > 1\}) \leq 2^{-1}$. Now choose $\epsilon_2 = 1/2$, we find $n_2 > n_1$ such that $\mu(\{x \in X : |f_{n_2}(x) - f(x)| > 1/2\}) \leq 2^{-2}$. Continuing in this manner, we find at the kth stage an $n_k > n_{k-1}$ such that $\mu(\{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}) \leq 2^{-k}$.

Proof(b) Let $A_k = \{x \in X : |f_{n_k}(x) - f(x)| > 1/k\}$ and $A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. By part (a) $\mu(A_k) \leq 2^{-k}$ and hence $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. By Borel-Cantelli Lemma (Exercise 6.9), we have $\mu(A) = 0$. For $x \notin A$, there exists $n \geq 1$ such that $x \notin \bigcup_{k=n}^{\infty} A_k$. This implies that $x \notin A_k$ for all $k \geq n$ and therefore $|f_{n_k}(x) - f(x)| \leq 1/k$ for all $k \geq n$. Thus, $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for all $x \in X \setminus A$. Since $\mu(X \setminus A) = 1$ we have that $f_{n_k} \to f$ μ a.e