Measure and Integration Quiz, 2016-17

1. Let $(X, \mathcal{A})$ be a measure space such that $\mathcal{A} = \sigma(\mathcal{G})$, where $\mathcal{G}$ is a collection of subsets of X such that $\emptyset \in \mathcal{G}$. Show that for any $A \in \mathcal{A}$ there exists a countable collection $\mathcal{G}_A \subseteq \mathcal{G}$ such that $A \in \sigma(\mathcal{G}_A)$. (2.5 pts.)

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(f_n)_n \subset \mathcal{M}^+(\mathcal{A})$ a sequence of non-negative real-valued measurable functions such that $\lim_{n\to\infty} f_n = f$ for some non-negative measurable function f. Assume that
$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu < \infty,$$
and let $A \in \mathcal{A}$.
   (i) Show that
$$\int \mathbf{1}_A f \, d\mu \geq \limsup_{n\to\infty} \int \mathbf{1}_A f_n \, d\mu.$$
   (Hint: apply Fatou's lemma to the sequence $g_n = f_n - \mathbf{1}_A f_n$.) (2.5 pts.)
   (ii) Prove that
$$\int \mathbf{1}_A f \, d\mu = \lim_{n\to\infty} \int \mathbf{1}_A f_n \, d\mu.$$
   (1 pt.)

3. Let $(X, \mathcal{A}, \mu)$ be a probability space (so $\mu(X) = 1$), and $T : X \to X$ an $\mathcal{A}/\mathcal{A}$ measurable function satisfying the following two properties:
   (a) $\mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{A}$,
   (b) if $A \in \mathcal{A}$ is such that $A = T^{-1}(A)$, then $\mu(A) \in \{0, 1\}$.
   The n-fold composition of T with itself is denoted by $T^n = T \circ T \circ \cdots \circ T$, and T^{-n} is the inverse image of the function T^n.
   (i) Let $B \in \mathcal{A}$ be such that $\mu(B \Delta T^{-1}(B)) = 0$. Prove that $\mu(B \Delta T^{-n}(B)) = 0$ for all $n \geq 1$. (Hint: note that $E \Delta F = (E \cap F^c) \cup (F \cap E^c)$, and that in any measure space one has $\mu(E \Delta F) \leq \mu(E \Delta G) + \mu(G \Delta F)$, justify the last statement) (1 pt.)
   (ii) Let $B \in \mathcal{A}$ be such that $\mu(B \Delta T^{-1}(B)) = 0$, and assume $\mu(B) > 0$. Define $C = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} T^{-n}(B)$. Prove that C satisfies $\mu(C) > 0$, and $T^{-1}(C) = C$. Conclude that $\mu(C) = 1$. (1.5 pts.)
   (iii) Let B and C be as in part (ii), show that
$$B \Delta C \subseteq \bigcup_{n=1}^{\infty} (T^{-n}(B) \Delta B).$$
   Conclude that $\mu(B \Delta C) = 0$, and $\mu(B) = 1$. (1.5 pts.)