Measure and Integration Solutions Quiz, 2016-17

1. Let $(X, \mathcal{A})$ be a measure space such that $\mathcal{A} = \sigma(\mathcal{G})$, where $\mathcal{G}$ is a collection of subsets of X such that $\emptyset \in \mathcal{G}$. Show that for any $A \in \mathcal{A}$ there exists a countable collection $\mathcal{G}_A \subseteq \mathcal{G}$ such that $A \in \sigma(\mathcal{G}_A)$. (2.5 pts.)

   Proof We apply *the good set principle.* Let

   $$\mathcal{B} = \{A \in \mathcal{A} : \exists \mathcal{G}_A \subseteq \mathcal{G} \text{ countable with } A \in \sigma(\mathcal{G}_A)\}.$$

   Clearly $\mathcal{B} \subseteq \mathcal{A}$. To prove the reverse containment, we show that $\mathcal{B}$ is a σ-algebra containing $\mathcal{G}$. For A in $\mathcal{G}$, let $\mathcal{G}_A = \{A\}$, clearly $\mathcal{G}_A \subseteq \mathcal{G}$ is countable and $A \in \sigma(\mathcal{G}_A) = \{\emptyset, X, A, A^c\}$, so $\mathcal{G} \subseteq \mathcal{B}$. Note that $\emptyset \in \mathcal{B}$, since $\emptyset \in \mathcal{G}$. Now, let $A \in \mathcal{B}$, and $\mathcal{G}_A \subseteq \mathcal{G}$ countable with $A \in \sigma(\mathcal{G}_A)$. Since $\sigma(\mathcal{G}_A)$ is a σ-algebra, we have $A^c \in \sigma(\mathcal{G}_A)$. Taking $\mathcal{G}_{A^c} = \mathcal{G}_A$ we see that $A^c \in \mathcal{B}$. Finally, let $(A_n)_n$ be a countable collection in $\mathcal{B}$. For each $n \geq 1$, there exists a countable collection $\mathcal{G}_{A_n} \subseteq \mathcal{G}$ such that $A_n \in \sigma(\mathcal{G}_{A_n}) \subseteq \sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{G}_{A_n}\Big)$. Now $\bigcup_{n=1}^{\infty} \mathcal{G}_{A_n} \subseteq \mathcal{G}$ is countable and $\bigcup_{n=1}^{\infty} A_n \in \sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{G}_{A_n}\Big)$, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$, and $\mathcal{B}$ is a σ-algebra hence $\mathcal{A} = \mathcal{B}$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $(f_n)_n \subset \mathcal{M}^+(\mathcal{A})$ a sequence of non-negative real-valued measurable functions such that $\lim_{n\to\infty} f_n = f$ for some non-negative measurable function f. Assume that

   $$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu < \infty,$$

   and let $A \in \mathcal{A}$.

   (i) Show that

   $$\int \mathbf{1}_A f \, d\mu \geq \limsup_{n\to\infty} \int \mathbf{1}_A f_n \, d\mu.$$

   (Hint: apply Fatou's lemma to the sequence $g_n = f_n - \mathbf{1}_A f_n$.) (2.5 pts.)

   (ii) Prove that

   $$\int \mathbf{1}_A f \, d\mu = \lim_{n\to\infty} \int \mathbf{1}_A f_n \, d\mu.$$

   (1 pt.)

   Proof(i) Let $g_n = f_n - \mathbf{1}_A f_n$, then by hypothesis

   $$f - \mathbf{1}_A f = \lim_{n\to\infty} g_n = \liminf_{n\to\infty} g_n.$$

   Since

   $$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu,$$

exists, then by Fatou's Lemma, and the linearity of the integral we have

$$\int f\,d\mu - \int \mathbf{1}_A f\,d\mu \leq \liminf_{n\to\infty} \int g_n\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu - \limsup_{n\to\infty} \int \mathbf{1}_A f_n\,d\mu.$$

The above gives

$$\int f\,d\mu - \int \mathbf{1}_A f\,d\mu \leq \int f\,d\mu - \limsup_{n\to\infty} \int \mathbf{1}_A f_n\,d\mu.$$

Subtracting $\int f\,d\mu < \infty$ from both sides leads to

$$\int \mathbf{1}_A f\,d\mu \geq \limsup_{n\to\infty} \int \mathbf{1}_A f_n\,d\mu.$$

Proof(ii) By Fatou's Lemma, we have

$$\int \mathbf{1}_A f\,d\mu = \int \lim_{n\to\infty} \mathbf{1}_A f_n\,d\mu \leq \liminf_{n\to\infty} \int \mathbf{1}_A f_n\,d\mu.$$

Combining with part (a), we have

$$\int \mathbf{1}_A f\,d\mu = \lim_{n\to\infty} \int \mathbf{1}_A f_n\,d\mu.$$

3. Let $(X, \mathcal{A}, \mu)$ be a probability space (so $\mu(X) = 1$), and $T : X \to X$ an $\mathcal{A}/\mathcal{A}$ measurable function satisfying the following two properties:

   (a) $\mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{A}$,

   (b) if $A \in \mathcal{A}$ is such that $A = T^{-1}(A)$, then $\mu(A) \in \{0, 1\}$.

   The n-fold composition of T with itself is denoted by $T^n = T \circ T \circ \cdots \circ T$, and T^{-n} is the inverse image of the function T^n.

   (i) Let $B \in \mathcal{A}$ be such that $\mu(B\Delta T^{-1}(B)) = 0$. Prove that $\mu(B\Delta T^{-n}(B)) = 0$ for all $n \geq 1$. (Hint: note that $E\Delta F = (E \cap F^c) \cup (F \cap E^c)$, and that in any measure space one has $\mu(E\Delta F) \leq \mu(E\Delta G) + \mu(G\Delta F)$, justify the last statement) (1 pt.)

   (ii) Let $B \in \mathcal{A}$ be such that $\mu(B\Delta T^{-1}(B)) = 0$, and assume $\mu(B) > 0$. Define $C = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} T^{-n}(B)$. Prove that C satisfies $\mu(C) > 0$, and $T^{-1}(C) = C$. Conclude that $\mu(C) = 1$. (1.5 pts.)

   (iii) Let B and C be as in part (ii), show that

   $$B\Delta C \subseteq \bigcup_{n=1}^{\infty} (T^{-n}(B)\Delta B).$$

   Conclude that $\mu(B\Delta C) = 0$, and $\mu(B) = 1$. (1.5 pts.)

Proof (i) We first show that for any three sets $E, F, G \in \mathcal{A}$ one has $\mu(E\Delta F) \leq \mu(E\Delta G) + \mu(G\Delta F)$.

Note that

$$E\cap F^c = \Big(E\cap F^c\cap G\Big)\cup\Big(E\cap F^c\cap G^c\Big) \subseteq (F^c\cap G)\cup(E\cap G^c) \subseteq (F\Delta G)\cup(E\Delta G).$$

Similarly,

$$F\cap E^c = \Big(F\cap E^c\cap G\Big)\cup\Big(F\cap E^c\cap G^c\Big) \subseteq (E^c\cap G)\cup(F\cap G^c) \subseteq (E\Delta G)\cup(F\Delta G).$$

Thus,

$$E\Delta F \subseteq (E\Delta G)\cup(F\Delta G).$$

By monotonicity and subadditivity of measures,

$$\mu(E\Delta F) \leq \mu(E\Delta G) + \mu(G\Delta F).$$

Note that inverse images respect all set operations, so by property (a) we have for any $n \geq 1$,

$$\mu\Big(T^{-1}(B)\Delta T^{-(n+1)}(B)\Big) = \mu\Big(B\Delta T^{-n}(B)\Big).$$

The proof is done by induction on n. By hypothesis the result is true for $n = 1$. Assume it is true for n, i.e. $\mu\Big(B\Delta T^{-n}(B)\Big) = 0$, we show it is true for $n+1$. By part (a),

$$\begin{aligned}
\mu\Big(B\Delta T^{-(n+1)}(B)\Big) &\leq \mu\Big(B\Delta T^{-1}(B)\Big) + \mu\Big(T^{-1}(B)\Delta T^{-n}(B)\Big) \\
&= \mu\Big(B\Delta T^{-1}(B)\Big) + \mu\Big(B\Delta T^{-n}(B)\Big) = 0.
\end{aligned}$$

The last equality follows from the induction hypothesis and our initial assumption on B.

Proof (ii) Clearly $C \in \mathcal{A}$. We first show that $\mu(C) > 0$. Let $C_m = \bigcup_{n=m}^{\infty} T^{-n}(B)$, note that C_m is a decreasing sequence, and $C = \bigcap_{m=1}^{\infty} C_m$. By property (a) and monotonicity of measures, we have for each $m \geq 1$,

$$\mu(C_m) \geq \mu\Big(T^{-m}(B)\Big) = \mu(B) > 0.$$

Thus by Theorem 4.4 (iii'),

$$\mu(C) = \lim_{m\to\infty} \mu(C_m) \geq \mu(B) > 0.$$

Since $(C_m)_m$ is a decreasing sequence, then

$$C = \bigcap_{m=1}^{\infty} C_m = \bigcap_{m=2}^{\infty} C_m = T^{-1}(C).$$

By property (b), and the fact $\mu(C) > 0$, we have that $\mu(C) = 1$.

Proof (iii)

$$\begin{aligned}
B\Delta C &= \Big(B\cap\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}T^{-n}(B^c)\Big)\cup\Big(B^c\cap\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}T^{-n}(B)\Big)\\
&\subseteq \Big(B\cap\bigcup_{m=1}^{\infty}T^{-m}(B^c)\Big)\cup\Big(B^c\cap\bigcup_{n=1}^{\infty}T^{-n}(B)\Big)\\
&= \bigcup_{n=1}^{\infty}\Big(B\cap T^{-n}(B^c)\Big)\cup\Big(B^c\cap T^{-n}(B)\Big)\\
&= \bigcup_{n=1}^{\infty}\Big(B\Delta T^{-n}(B)\Big).
\end{aligned}$$

By monotonicity and σ-subadditivity of measures, it follows from part (i) that

$$\mu(B\Delta C)\leq\sum_{n=1}^{\infty}\mu(B\Delta T^{-n}(B))=0.$$

Thus, $\mu(B\Delta C)=0$. This imples that $\mu(B\cap C^c)=\mu(C\cap B^c)=0$, so

$$\mu(B)=\mu(B\cap C)=\mu(C)=1.$$