Measure and Integration Solutions Quiz Extra, 2016-17

1. Let X be a set and $\mathcal{F}$ a collection of real valued functions on X satisfying the following properties:

   (i) $\mathcal{F}$ contains the constant functions,

   (ii) if $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$, then $f+g, fg, cf \in \mathcal{F}$,

   (ii) if $f_n \in \mathcal{F}$, and $f = \lim_{n\to\infty} f_n$, then $f \in \mathcal{F}$.

   For $A \subseteq X$, denote by $\mathbf{1}_A$ the indicator function of A, i.e.

   $$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A, \\ 0, & x \notin A. \end{cases}$$

   Show that the collection $\mathcal{A} = \{A \subseteq X : \mathbf{1}_A \in \mathcal{F}\}$ is a σ-algebra. (2.5 pts.)

   Proof Since $\mathbf{1}_X(x) = 1$ for all $x \in X$, then $\mathbf{1}_X$ is the constant function 1 so by property (i), $\mathbf{1}_X \in \mathcal{F}$ and hence $X \in \mathcal{A}$. Now, let $A \in \mathcal{A}$, then $\mathbf{1}_A \in \mathcal{F}$. Since $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$, then by property (ii) we have $\mathbf{1}_{A^c} \in \mathcal{F}$ so $A^c \in \mathcal{A}$. Finally, consider a sequence (A_n) with $A_n \in \mathcal{A}$, then $\mathbf{1}_{A_n} \in \mathcal{F}$ for all n, and by the above $\mathbf{1}_{A_n^c} \in \mathcal{F}$ for all n. By property (ii), we have $\mathbf{1}_{A_1^c}\mathbf{1}_{A_2^c}\cdots\mathbf{1}_{A_n^c} \in \mathcal{F}$, hence $\mathbf{1}_{\cup_{m=1}^n A_m} = 1 - \mathbf{1}_{\cap_{m=1}^n A_m^c} = 1 - \mathbf{1}_{A_1^c}\mathbf{1}_{A_2^c}\cdots\mathbf{1}_{A_n^c} \in \mathcal{F}$ for all n. Since $\mathbf{1}_{\cup_{n=1}^\infty A_n} = \lim_{n\to\infty} \mathbf{1}_{\cup_{m=1}^n A_m}$, then by property (iii) $\mathbf{1}_{\cup_{n=1}^\infty A_n} \in \mathcal{F}$, so $\bigcup_{n=1}^\infty A_n \in \mathcal{A}$. Thus $\mathcal{A}$ is a σ-algebra.

2. Let $(X, \mathcal{D}, \mu)$ be a measure space, and let $\overline{\mathcal{D}}^\mu$ be the completion of the σ-algebra $\mathcal{D}$ with respect to the measure μ (see exercise 4.13, p.29). We denote by $\overline{\mu}$ the extension of the measure μ to the σ-algebra $\overline{\mathcal{D}}^\mu$. Suppose $f : X \to X$ is a function such that $f^{-1}(B) \in \mathcal{D}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{D}$. Show that $f^{-1}(\overline{B}) \in \overline{\mathcal{D}}^\mu$ and $\overline{\mu}(f^{-1}(\overline{B})) = \overline{\mu}(\overline{B})$ for all $\overline{B} \in \overline{\mathcal{D}}^\mu$. (2.5 pts.)

   Proof: Let $\overline{B} \in \overline{\mathcal{D}}^\mu$, then there exist $A, B \in \mathcal{D}$ such that $A \subseteq \overline{B} \subseteq B$, $\mu(B \setminus A) = 0$ and $\overline{\mu}(\overline{B}) = \mu(A)$. Then, $f^{-1}(A)$, $f^{-1}(B) \in \mathcal{D}$ satisfy $f^{-1}(A) \subseteq f^{-1}(\overline{B}) \subseteq f^{-1}(B)$ and $\mu(f^{-1}(B) \setminus f^{-1}(A)) = \mu(f^{-1}(B \setminus A)) = \mu(B \setminus A) = 0$. Thus, $f^{-1}(\overline{B}) \in \overline{\mathcal{D}}^\mu$ and $\overline{\mu}(f^{-1}(\overline{B})) = \mu(f^{-1}(A) = \mu(A) = \overline{\mu}(\overline{B})$.

3. Consider the measure space $([0,1]\mathcal{B}([0,1]), \lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to $[0,1]$, and λ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1, \cdots, E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x \in [0,1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n \leq m$. Show that there exists a $j \in \{1, \cdots, m\}$ such that $\lambda(E_j) \geq \dfrac{n}{m}$. (2.5 pts.)

Proof: By hypothesis, for any $x \in [0,1]$ we have $\sum_{j=}^{m} \mathbf{1}_{E_j}(x) \geq n$. Assume for the sake of getting a contradiction that $\lambda(E_j) < \frac{n}{m}$ for all $1 \leq j \leq m$. Then,

$$n = \int_{[0,1]} n \, d\lambda \leq \int \sum_{j=}^{m} \mathbf{1}_{E_j}(x) \, d\lambda = \sum_{j=1}^{m} \lambda(E_j) < \sum_{j=}^{m} \frac{n}{m} = n,$$

a contradiction. Hence, there exists $j \in \{1, \cdots, m\}$ such that $\lambda(E_j) \geq \frac{n}{m}$.

4. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$. Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f \, d\mu \leq \int_E f \, d\nu$. (2.5 pts.)

Proof Suppose first that $f = 1_A$ is the indicator function of some set $A \in \mathcal{B}$. Then

$$\int_E f \, d\mu = \mu(A) \leq \nu(A) = \int_E f \, d\nu.$$

Suppose now that $f = \sum_{k=1}^{n} \alpha_k 1_{A_k}$ is a non-negative measurable simple function. Then,

$$\int_E f \, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) \leq \sum_{k=1}^{n} \alpha_k \nu(A_k) = \int_E f \, d\nu.$$

Finally, let f be a non-negative measurable function, then there exists a sequence of non-negative measurable simple functions f_n such that $f_n \uparrow f$. By Beppo-Levi,

$$\int_E f \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu \leq \lim_{n \to \infty} \int_E f_n \, d\nu = \int_E f \, d\nu.$$