Measure and Integration: Retake Final 2016-17

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra and λ is Lebesgue measure. Let $B \in \mathcal{B}(\mathbb{R})$ be such that $0 < \lambda(B) < \infty$, and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \lambda\Big(B \cap (-\infty, x]\Big).$$

(a) Prove that g is a uniformly continuous function. (1 pt)

(b) Show that for any $\alpha \in (0, \lambda(B))$ there exists a Borel measurable subset C_α of B such that $\lambda(C_\alpha) = \alpha$. (1 pt)

(2) Let $(X, \mathcal{A}, \mu)$ be a **finite** measure space, and $f \in \mathcal{M}(\mathcal{A})$ such that $f > 0$ μ a.e. Define

$$D = \{x \in X : f(x) > 0\} \text{ and } D_n = \{x \in D : f(x) \geq 1/n\},\ n \geq 1.$$

(a) Show that for every $\epsilon > 0$ there exists $n_0 \geq 1$ such that $\mu(D \setminus D_{n_0}) < \epsilon$. (1 pt)

(b) Show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $E \in \mathcal{A}$ with $\mu(E) \geq \epsilon$, one has $\displaystyle\int_E f\, d\mu \geq \delta$. (1 pt)

(3) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $p \in [1, \infty)$.

(a) Let $f, f_n \in \mathcal{L}^p(\mu)$ satisfy $\lim\limits_{n\to\infty} ||f_n - f||_p = 0$, and $g, g_n \in \mathcal{M}(\mathcal{A})$ satisfy $\lim\limits_{n\to\infty} g_n = g$ μ a.e. Assume that $|g_n| \leq M$, where $M > 0$ is a real number. Show that $\lim\limits_{n\to\infty} ||f_n g_n - fg||_p = 0$. (1 pt)

(b) Assume that $\mu(X) < \infty$, and $u_n, u, w_n, w \in \mathcal{M}(\mathcal{A})$ such that $u_n \xrightarrow{\mu} u$, and $w_n \xrightarrow{\mu} w$ (i.e. convergence is in measure). Assume further that $|w| \leq M$ and $|u_n| \leq M$ for all n, where M is some positive real number. Show that $u_n w_n \xrightarrow{\mu} uw$. (1 pt)

(4) Consider the function $u : (1, 2) \times \mathbb{R} \to \mathbb{R}$ given by $u(t, x) = e^{-tx^2} \cos x$. Let λ denotes Lebesgue measure on $\mathbb{R}$, show that the function $F : (1, 2) \to \mathbb{R}$ given by $F(t) = \displaystyle\int_{\mathbb{R}} e^{-tx^2} \cos x\, d\lambda(x)$ is differentiable. (1 pt)

(5) Let $(X, \mathcal{A}, \mu_1)$ and $(Y, \mathcal{B}, \nu_1)$ be σ-finite measure spaces. Suppose $f \in \mathcal{L}^1(\mu_1)$ and $g \in \mathcal{L}^1(\nu_1)$ are non-negative. Define measures μ_2 on $\mathcal{A}$ and ν_2 on $\mathcal{B}$ by

$$\mu_2(A) = \int_A f\, d\mu_1 \text{ and } \nu_2(B) = \int_B g\, d\nu_1,$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(a) For $D \in \mathcal{A} \otimes \mathcal{B}$ and $y \in Y$, let $D_y = \{x \in X : (x, y) \in D\}$. Show that if $\mu_1(D_y) = 0$ ν_1 a.e., then $\mu_2(D_y) = 0$ ν_2 a.e. (1 pt)

(b) Show that if $D \in \mathcal{A} \otimes \mathcal{B}$ is such that $(\mu_1 \times \nu_1)(D) = 0$ then $(\mu_2 \times \nu_2)(D) = 0$. (1 pt)

(c) Show that for every $D \in \mathcal{A} \otimes \mathcal{B}$ one has

$$(\mu_2 \times \nu_2)(D) = \int_D f(x) g(y)\, d(\mu_1 \times \nu_1)(x, y).$$

(1 pt)