Measure and Integration: Solutions Final Exam 2020-21

(1) Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in \mathcal{L}^1(\mu)$. Define the mesure ν on $\mathcal{A}$ by $\nu(A) = \int_A |u| \, d\mu$. Prove that for any $v \in \mathcal{L}^1(\nu)$, one has

$$\int v \, d\nu = \int |u| v \, d\mu.$$

(1.5 pts)

Proof: We use the standard argument. We assume first that $v = \mathbb{I}_A$ for some $A \in \mathcal{A}$. Then

$$\int v \, d\nu = \int \mathbb{I}_A \, d\nu = \nu(A) = \int |u| \mathbb{I}_A \, d\mu = \int |u| v \, d\mu.$$

Assume $v = \sum_{i=0}^n a_i \mathbb{I}_{A_i}$ is a simple function in standard form, with $A_i \in \mathcal{A}$. By linearity of the integral, we have

$$\begin{aligned}\int v \, d\nu &= \int \sum_{i=0}^n a_i \mathbb{I}_{A_i} \, d\nu \\ &= \sum_{i=0}^n a_i \mathbb{I}_{A_i} \int \mathbb{I}_{A_i} \, d\nu \\ &= \sum_{i=0}^n a_i \mathbb{I}_{A_i} \int |u| \mathbb{I}_{A_i} \, d\mu \\ &= \int |u| \sum_{i=0}^n a_i \mathbb{I}_{A_i} \, d\mu \\ &= \int |u| v \, d\mu.\end{aligned}$$

Now assume that $v \geq 0$, then by Theorem 8.8 there exists an increasing sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{E}^+(\mathcal{A})$ with $v = \sup_{n \geq 1} f_n$. By Beppo-Lévi (applied twice) and the above, we have

$$\int v \, d\nu = \sup_{n \geq 1} \int f_n \, d\nu = \sup_{n \geq 1} \int |u| f_n \, d\mu = \int |u| \sup_{n \geq .1} f_n \, d\mu = \int |u| v \, d\mu.$$

Finally, for a general $v \in \mathcal{L}^1(\nu)$, we write $v = v^+ - v^-$ and note that $v^+, v^- \in \mathcal{L}^1(\mu)$. By the linearity of the integral and the above verifications for non-negative integrable functions, we have

$$\int v \, d\nu = \int v^+ \, d\nu - \int v^- \, d\nu = \int |u| v^+ \, d\mu - \int |u| v^- \, d\mu = \int |u| (v^+ - v^-) \, d\mu = \int |u| v \, d\mu.$$

(2) Consider the measure space $\big((0,1), \mathcal{B}((0,1)), \lambda\big)$, where $\mathcal{B}((0,1))$ is the Borel σ-algebra restricted to the interval $(0,1)$ and λ is the restriction of Lebesgue measure to $(0,1)$. Let $u \in \mathcal{L}^2(\lambda)$ be **non-negative** and **monotonically increasing**.

(a) Prove that for any $x \in (0,1)$, $\inf_{n \geq 1} u(x^n) = \inf_{y \in (0,1)} u(y)$. (0.5 pt)

(b) Let $w_n(x) = x \cdot u(x^n)$, $n \geq 1$. Prove that $w_n \in \mathcal{L}^2(\lambda)$ for all $n \geq 1$, and that $\lim_{n \to \infty} \|w_n(x)\|_2 = \inf_{y \in (0,1)} u(y) \cdot \dfrac{\sqrt{3}}{3}$. (2 pts)

(c) Prove that $\lim_{n \to \infty} \int_{(0,1)} x^n e^{x/n} u(x) \, d\lambda(x) = 0$. (1 pt)

Proof(a): Note that for any $x \in (0,1)$, the sequence $(x^n)_{n\in\mathbb{N}}$ decreases to 0, so that

$$(0,1) = \dot{\bigcup_{n=1}^{\infty}} [x^n, x^{n-1}).$$

Since u is monotonically increasing, we have

$$\inf_{y\in(0,1)} u(y) = \inf_{n\geq 1} \inf_{y\in[x^n,x^{n-1})} u(y) = \inf_{n\geq 1} u(x^n).$$

Proof(b): The function $x \to x^n$ is Borel measurable since it is continuous, and since u is Borel measurable it follows that w_n is Borel measurable. Since u is monotonically increasing, then the same holds for u^2. Now for any $x \in (0,1)$ we have $x^n < x < 1$, hence $0 \leq w_n^2(x) \leq u^2(x)$ for all x. Since $u^2 \in \mathcal{L}^1(\lambda)$, it follows that $w_n^2 \in \mathcal{L}^1(\lambda)$ for all n, i.e . $w_n \in \mathcal{L}^2(\lambda)$ for all n. For any $x \in (0,1)$ we have

$$\lim_{n\to\infty} w_n^2(x) = \lim_{n\to\infty} x^2 \cdot u^2(x^n) = x^2 \cdot \inf_{n\geq 1} u^2(x^n) = x^2 \Big(\inf_{y\in(0,1)} u(y) \Big)^2.$$

By Lebesgue Dominated Convergence Theorem, the fact that the function $f(x) = x^2$ is Riemann-integrable on the interval $[0,1]$ and Theorem 11.2(ii), we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{(0,1)} w_n^2(x)\, d\lambda(x) &= \int_{(0,1)} \lim_{n\to\infty} w_n^2(x)\, d\lambda(x) \\
&= \Big(\inf_{y\in(0,1)} u(y) \Big)^2 \int_{(0,1)} x^2\, d\lambda(x) \\
&= \Big(\inf_{y\in(0,1)} u(y) \Big)^2 \int_{[0,1]} x^2\, d\lambda(x) \\
&= \Big(\inf_{y\in(0,1)} u(y) \Big)^2 (R) \int_0^1 x^2\, dx \\
&= \frac{1}{3} \Big(\inf_{y\in(0,1)} u(y) \Big)^2.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|w_n\|_2 = \lim_{n\to\infty} \Big(\int_{(0,1)} w_n^2(x)\, d\lambda(x) \Big)^{1/2} = \inf_{y\in(0,1)} u(y) \cdot \frac{\sqrt{3}}{3}.$$

Proof(c): First note that since $\lambda((0,1)) = 1$ and $\|u\|_2 < \infty$. By Hölder's inequality,

$$\int |u|\, d\lambda = \int |u \cdot 1| d\lambda \leq \|u\|_2 \|1\|_2 = \|u\|_2 < \infty.$$

Thus $u \in \mathcal{L}^1(\lambda)$. For each $x \in (0,1)$ and for every $n \geq 1$, we have $0 \leq x^n e^{x/n} u(x) < eu(x)$. Set $v_n(x) = x^n e^{x/n} u(x)$. Since $eu \in \mathcal{L}^1(\lambda)$, then $v_n \in \mathcal{L}^1(\lambda)$ for all n. Furthermore, $\lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} x^n e^{x/n} u(x) = 0$ for all $x \in (0,1)$ (note the $u(x) < \infty$). By Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{(0,1)} x^n e^{x/n} u(x)\, d\lambda(x) = \int_{(0,1)} \lim_{n\to\infty} x^n e^{x/n} u(x)\, d\lambda(x) = 0.$$

(3) Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 < p < \infty$. Suppose $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p(\mu)$ with $\|u_n\|_p \leq \dfrac{1}{2p+1}$ for $n \geq 1$. Prove that $\Big| \sum_{n=1}^{\infty} \Big(\frac{u_n}{n} \Big)^p \Big| < \infty$ μ a.e. (2 pts)

Proof: By Corollary 11.6, it is enough to show that $\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\right)^p \in \mathcal{L}^1(\mu)$, equivalently $\left|\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\right)^p\right| \in \mathcal{L}^1(\mu)$. By Corollary 9.9 and the fact that $1 < p < \infty$ we have

$$\begin{aligned}\int \sum_{n=1}^{\infty}\left(\frac{|u_n|}{n}\right)^p d\lambda &= \sum_{n=1}^{\infty}\int \frac{|u_n|^p}{n^p} d\lambda \\ &\leq \left(\frac{1}{2p+1}\right)^p \sum_{n=1}^{\infty}\frac{1}{n^p} \\ &< \infty,\end{aligned}$$

implying that $\sum_{n=1}^{\infty}\left(\frac{|u_n|}{n}\right)^p \in \mathcal{L}^1(\mu)$. Since

$$\left|\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\right)^p\right| \leq \sum_{n=1}^{\infty}\left(\frac{|u_n|}{n}\right)^p,$$

it follows that $\left|\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\right)^p\right| \in \mathcal{L}^1(\mu)$ and therefore $\left|\sum_{n=1}^{\infty}\left(\frac{u_n}{n}\right)^p\right| < \infty$ μ a.e.

(4) Consider the product space $\Big([1,2]\times[0,\infty), \mathcal{B}([1,2])\otimes\mathcal{B}([0,\infty)), \lambda\times\lambda\Big)$, where λ is Lebesgue measure restricted to the appropriate space. Consider the fuction $f : [1,2]\times[0,\infty) \to [0,\infty)$ defined by $f(x,t) = e^{-2xt}\mathbb{I}_{(0,\infty)}(t)$.

(a) Prove that $f \in \mathcal{L}^1(\lambda\times\lambda)$. (2 pts)

(b) Prove that $\int_{(0,\infty)}(e^{-2t} - e^{-4t})\frac{1}{t}d\lambda(t) = \ln(2)$. (1 pt)

Proof (a) Since both the functions $(x,t) \to e^{-2xt}$ and $(x,t) \to \mathbb{I}_{(0,\infty)}(t)$ are measurable, it follows that $f \in \mathcal{M}^+(\mathcal{B}([1,2]))\otimes\mathcal{B}([0,\infty))$. For each fixed $x \in [1,2]$, the function $t \to e^{-2xt}$ is positive measurable and the improper Riemann integrable on $[0,\infty)$ exists, so that

$$\int_{[0,\infty)} f(x,t)d\lambda(t) = \int_{(0,\infty)} e^{-2xt}d\lambda(t) = \int_{[0,\infty)} e^{-2xt}d\lambda(t) = (R)\int_0^{\infty} e^{-2xt}dt = \frac{1}{2x}.$$

The second equality follows from the fact that $\lambda(\{0\}) = 0$. Furthermore, the function $x \to \frac{1}{2x}$ is measurable and Riemann integrable on $[1,2]$, thus

$$\int_{[1,2]}\int_{[0,\infty)} f(x,t)d\lambda(t)\,d\lambda(x) = \int_{[1,2]}\frac{1}{2x}\,d\lambda(x) = (R)\int_1^2 \frac{1}{2x}\,dx = \frac{\ln(2)}{2} < \infty.$$

Thus, by Fubini's Theorem $f \in \mathcal{L}^1(\lambda\times\lambda)$ and $\int_{[1,2]\times[0,\infty)} f\,d(\lambda\times\lambda) = \frac{\ln(2)}{2}$.

Proof (b) Note that by part (a), we see that

$$\int_{[1,2]\times[0,\infty)} f(x,t)\,d(\lambda\times\lambda)(x,t) = \int_{[1,2]\times(0,\infty)} e^{-2xt}\,d(\lambda\times\lambda)(x,t) = \frac{\ln(2)}{2}.$$

By Toneli's Theorem (or Fubini) this implies that

$$\int_{[1,2]\times[0,\infty)} f\,d(\lambda\times\lambda) = \int_{(0,\infty)}\int_{[1,2]} e^{-2xt}d\lambda(x)\,d\lambda(t) = \int_{[1,2]}\int_{(0,\infty)} e^{-2xt}d\lambda(t)\,d\lambda(x) = \frac{\ln 2}{2}.$$

However,

$$\int_{(0,\infty)}\int_{[1,2]} e^{-2xt}d\lambda(x)\,d\lambda(t) = \int_{(0,\infty)}\Big((R)\int_{[1,2]} e^{-2xt}dx\Big)\,d\lambda(t) = \int_{(0,\infty)}(e^{-2t} - e^{-4t})\frac{1}{2t}d\lambda(t).$$

Therefore, $\int_{(0,\infty)}(e^{-2t} - e^{-4t})\frac{1}{t}d\lambda(t) = \ln(2)$.