Measure and Integration: Solutions Practice Final Exam 2020-21

(1) Consider the measure space $[1,\infty),\mathcal{B}([1,\infty]),\lambda)$ where $\mathcal{B}([1,\infty])$ is the Borel σ-algebra and λ is the Lebesgue measure restricted to $[1,\infty)$. Show that

$$\lim_{n\to\infty}\int_{[1,\infty)}\frac{n\sin(x/n)}{x^3}\,d\lambda(x)=1.$$

(Hint: $\lim\limits_{x\to 0}\sin(x)/x=1$)

Proof: Let $u_n(x)=\dfrac{n\sin(x/n)}{x^3}$, then u_n is continuous on $[1,\infty)$ and hence is measurable. Note that $|\sin(y)|\leq y$ for all $y\geq 0$, hence $u_n(x)\leq 1/x^2$. Furthermore, $\lim\limits_{n\to\infty}u_n(x)=\dfrac{1}{x^2}$. Since the function $\dfrac{1}{x^2}$ is positive, measurable and the improper Riemann integral on $[1,\infty)$ exists, it follows it is Lebesgue integrable on $[1,\infty)$. By Lebesgue Dominated Convergence Theorem we have

$$\begin{aligned}\lim_{n\to\infty}\int_{[1,\infty)}\frac{n\sin(x/n)}{x^3}\,d\lambda(x) &= \int_{[1,\infty)}\lim_{n\to\infty}\frac{n\sin(x/n)}{x^3}\,d\lambda(x)\\ &= \int_{[1,\infty)}\frac{1}{x^2}\,d\lambda(x)\\ &= (R)\int_1^\infty\frac{1}{x^2}\,dx=1.\end{aligned}$$

(2) Let $(X,\mathcal{A},\mu)$ be a measure space, and $p,q\in(1,\infty)$ and $r\geq 1$ be such that $1/r=1/p+1/q$. Show that if $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$, then $fg\in\mathcal{L}^r(\mu)$ and $\|fg\|_r\leq\|f\|_p\,\|g\|_q$.

Proof: Let $p'=p/r$ and $q'=q/r$, since $1/r=1/p+1/q$ we have $1=1/p'+1/q'$. Suppose $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$, and set $F=f^r$ and $G=g^r$. Then,

$$\int|F|^{p'}\,d\mu=\int|f|^p\,d\mu<\infty,$$

and

$$\int|G|^{q'}\,d\mu=\int|g|^q\,d\mu<\infty.$$

Hence, $F\in\mathcal{L}^{p'}(\mu)$, and $G\in\mathcal{L}^{q'}(\mu)$. By Hölder's inequality we have $(fg)^r=FG\in\mathcal{L}^1(\mu)$, which implies $fg\in\mathcal{L}^r(\mu)$, and

$$\begin{aligned}\int|fg|^r\,d\mu &= \int|FG|\,d\mu\\ &\leq \Big(\int|F|^{p'}\,d\mu\Big)^{1/p'}\Big(\int|G|^{q'}\,d\mu\Big)^{1/q'}\\ &= \Big(\int|f|^p\,d\mu\Big)^{r/p}\Big(\int|g|^q\,d\mu\Big)^{r/q}.\end{aligned}$$

Hence,

$$\Big(\int|fg|^r\,d\mu)^{1/r}\leq\Big(\int|f|^p\,d\mu\Big)^{1/p}\Big(\int|g|^q\,d\mu\Big)^{1/q},$$

equivalently, $\|fg\|_r\leq\|f\|_p\|g\|_q$.

(3) Consider the function $u:(1,2)\times\mathbb{R}\to\mathbb{R}$ given by $u(t,x)=e^{-tx^2}\cos x$. Let λ denotes Lebesgue measure on $\mathbb{R}$, show that the function $F:(1,2)\to\mathbb{R}$ given by $F(t)=\int_{\mathbb{R}} e^{-tx^2}\cos x\,d\lambda(x)$ is differentiable.

Proof: We apply Theorem 12.5 (11.5 in edition 1). First for any fixed $t\in(1,2)$, we have

$$|e^{-tx^2}\cos x|\le e^{-tx^2}\le e^{-x^2/2}\in\mathcal{L}^1(\lambda)$$

since $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx=\sqrt{2\pi}$ implying $\int_{\mathbb{R}} e^{-tx^2}\,d\lambda(x)\le\int_{\mathbb{R}} e^{-x^2/2}\,d\lambda(x)=\sqrt{2\pi}$. Now for any fixed $x\in\mathbb{R}$, the function $t\to u(t,x)=e^{-tx^2}\cos x$ is differentiable with $\frac{\partial u}{\partial t}(t,x)=-x^2e^{-tx^2}\cos x$. Using the second and third term of the Taylor series expansion of e^{tx^2} one gets $e^{tx^2}\ge tx^2+t^2x^4/2$, and hemce

$$|\frac{\partial u}{\partial t}(t,x)|\le x^2e^{-tx^2}\le\frac{1}{t}\frac{1}{1+tx^2/2}\le\frac{2}{t}\frac{1}{1+tx^2}\le\frac{2}{1+x^2}.$$

It is easy to check that $\int_{-\infty}^{\infty}\frac{2}{1+x^2}\,dx=2\pi$, so $\int_{-\infty}^{\infty}\frac{2}{1+x^2}\,d\lambda(x)=2\pi$ implying that the function $w(x)=\frac{2}{1+x^2}\in\mathcal{L}^1(\lambda)$. Thus by Theorem 12.5 (11.5 in edition 1), the function $F(t)=\int_{\mathbb{R}} e^{-tx^2}\cos x\,d\lambda(x)$ is differentiable, and

$$\frac{dF}{dt}(t)=\int_{\mathbb{R}}\frac{\partial}{\partial t}(e^{-tx^2}\cos x)\lambda(x)=\int_{\mathbb{R}} -x^2e^{-tx^2}\cos x\,d\lambda(x).$$

(4) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Let $k,g\in\mathcal{L}^1(\lambda)$ and define $F:\mathbb{R}^2\to\mathbb{R}$, and $h:\mathbb{R}\to\overline{\mathbb{R}}$ by

$$F(x,y)=k(x-y)g(y).$$

(a) Show that F is measurable.
(b) Show that $F\in\mathcal{L}^1(\lambda\times\lambda)$, and

$$\int_{\mathbb{R}\times\mathbb{R}} F(x,y)d(\lambda\times\lambda)(x,y)=\Big(\int_{\mathbb{R}} k(x)d\lambda(x)\Big)\Big(\int_{\mathbb{R}} g(y)d\lambda(y)\Big).$$

Proof(a): To show measurablity of F, we first extend the domain of g to $\mathbb{R}^2$ as follows. Define $\overline{g}:\mathbb{R}^2\to\mathbb{R}$ by $\overline{g}(x,y)=g\circ\pi_2(x,y)=g(y)$. It is easy to see that $\overline{g}$ is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable. Moreover, the function $d:\mathbb{R}^2\to\mathbb{R}$ given by $d(x,y)=x-y$ is continuous hence $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable. Since

$$F(x,y)=k(x-y)g(y)=k\circ d(x,y)\overline{g}(x,y)$$

is the product of two $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable functions, it follows that F is $\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R})$ measurable.

Proof(b): Since Lebesgue measure is translation invariant, we have

$$\begin{aligned}\int\int|F(x,y)|\,d\lambda(x)d\lambda)(y) &= \int\int|k(x-y)||g(y)|\,d\lambda(x)d\lambda)(y)\\ &= \int\int|k(x)||g(y)|\,d\lambda(x)d\lambda)(y)\\ &= \int|k(x)|\,d\lambda(x)\int|g(y)|d\lambda)(y)<\infty.\end{aligned}$$

By Fubini's Theorem, this implies that F is $\lambda\times\lambda$ integrable, and

$$\begin{aligned}\int F(x,y)\,d(\lambda\times\lambda)(x,y) &= \int\int k(x-y)g(y)\,d\lambda(x)\,d\lambda(y)\\ &= \int\int k(x)g(y)\,d\lambda(x)d\lambda)(y)\\ &= \int k(x)\,d\lambda(x)\int g(y)d\lambda)(y).\end{aligned}$$

(5) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R}))$ is the Borel σ-algebra and λ is Lebesgue measure. Let $f\in\mathcal{L}^1(\lambda)$ and define for $h>0$, the function $f_h(x)=\frac{1}{h}\int_{[x,x+h]} f(t)\,d\lambda(t)$.

(a) Show that f_h is Borel measurable for all $h > 0$.
(b) Show that $f_h \in \mathcal{L}^1(\lambda)$ and $\|f_h\|_1 \le \|f\|_1$.

Proof (a): For $h > 0$, define $u_h(t,x) = \frac{1}{h}\mathbf{1}_{[x,x+h]}(t)f(t)$, then u_h is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ measurable. Applying Tonelli's Theorem (Theorem 13.8(ii)) to the positive and negative parts of the function u_h, we have that the functions

$$x \to \int u^+(t,x)\, d\lambda(t) = f_h^+(x), \text{ and } x \to \int u^-(t,x)\, d\lambda(t) = f_h^-(x)$$

are $\mathcal{B}(\mathbb{R})$ measurable. Hence, f_h is Borel measurable for all $h > 0$.

Proof(b): Note that

$$\int \int \frac{1}{h}\mathbb{I}_{[x,x+h]}(t)|f(t)|\, d\lambda(x)\, d\lambda(t) = \int \int \frac{1}{h}\mathbb{I}_{[t-h,t]}(x)|f(t)|\, d\lambda(x)\, d\lambda(t) = \int |f(t)|\, d\lambda(t) < \infty.$$

Hence, by Fubini's Theorem $f_h \in \mathcal{L}^1(\lambda)$ and

$$\int |f_h(x)|\, d\lambda(x) = \int \int \frac{1}{h}\mathbb{I}_{[x,x+h]}(t)|f(t)|\, d\lambda(x)\, d\lambda(t) = \int |f(x)|\, d\lambda(x) = \|f\|_1.$$

(6) Let $(X, \mathcal{A}, \mu)$ be a measure space, and $p \in [1, \infty)$. Let $f, f_n \in \mathcal{L}^p(\mu)$ satisfy $\lim_{n\to\infty} \|f_n - f\|_p = 0$, and $g, g_n \in \mathcal{M}(\mathcal{A})$ satisfy $\lim_{n\to\infty} g_n = g$ μ a.e. Assume that $|g_n| \le M$, where $M > 0$ is a real number. Show that $\lim_{n\to\infty} \|f_n g_n - fg\|_p = 0$.

Proof: We have

$$|f_n g_n - fg| = |f_n g_n - g_n f + g_n f - fg| \le |g_n|\,|f_n - f| + |f|\,|g_n - g|.$$

Hence,

$$\begin{aligned} |f_n g_n - fg|^p &\le 2^p\Big(|g_n|^p\,|f_n - f|^p + |f|^p\,|g_n - g|^p\Big) \\ &\le 2^p\Big(|M^p\,|f_n - f|^p + |f|^p\,|g_n - g|^p\Big). \end{aligned}$$

This gives

$$\int |f_n g_n - fg|^p\, d\mu \le 2^p M^p \int |f_n - f|^p\, d\mu + 2^p \int |f|^p |g_n - g|^p\, d\mu.$$

Since, $\lim_{n\to\infty} \|f_n - f\|_p = 0$ we have $\lim_{n\to\infty} 2^p M^p \int |f_n - f|^p\, d\mu = 0$. For the second term, we observe that $\lim_{n\to\infty} |f|^p |g_n - g|^p = 0$ μ a.e. and $|f|^p|g_n - g|^p \le |f|^p(2M)^p \in \mathcal{L}^1(\mu)$. Hence, by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} 2^p \int |f|^p |g_n - g|^p\, d\mu = 2^p \int \lim_{n\to\infty} |f|^p |g_n - g|^p\, d\mu = 0.$$

The above imply that $\lim_{n\to\infty} \|f_n g_n - fg\|_p = 0$.