Functionaalanalyse

Problem 1 (polarization identity, 4 pt). Let $(X, \langle\cdot,\cdot\rangle)$ be a Hermitian inner product space. (Hence $\mathbb{K} = \mathbb{C}$.) Show that

$$\langle x, y\rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\Big), \quad \forall x, y \in X.$$

Problem 2 (comparing ℓ^p-norms, 4 pt). Let $p \leq q \in [1, \infty)$. Show that

$$\ell^p \subseteq \ell^q,$$
$$\|x\|_q \leq \|x\|_p, \quad \forall x \in \ell^p.$$

Problem 3 (quotient seminorm, 6 pt). Let $(X, \|\cdot\|)$ be a semi-normed vector space and $Y \subseteq X$ a linear subspace. Prove that the following map is a seminorm:

$$\|\cdot\|^Y : X/Y \to [0, \infty),$$
$$\|x + Y\|^Y := \inf_{y \in Y} \|x - y\|.$$

Remark: You do not need to prove that X/Y is a vector space.

Problem 4 (integration bounded, 5 pt). We equip $C[0,1]$ with the supremum norm $\|\cdot\|_\infty$. Let $y \in C[0,1]$. Show that the operator

$$T : C[0,1] \to \mathbb{K}, \quad Tx := \int_0^1 x(t)y(t)\,dt,$$

is bounded and calculate its operator norm.

Hint: Prove that for a certain constant $C \geq 0$ we have $\|T\| \leq C$ and $\|T\| \geq C - \varepsilon$, for every $\varepsilon > 0$.

Problem 5 (equivalence of norms, 3 pt). Let $\|\cdot\|$ and $|||\cdot|||$ be complete norms on a vector space X, such that there exists $C \in \mathbb{R}$ satisfying

$$\|x\| \leq C|||x|||, \quad \forall x \in X.$$

Show that $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

Hint: Use a result from the lecture.

Problem 6 (dual space of ℓ^∞/c_0, 4 pt). Prove that the dual space of ℓ^∞/c_0, equipped with the quotient norm, is nonzero.

Problem 7 (Hilbert space reflexive, 7 pt). Prove that every Hilbert space H is reflexive.

Hint: Relate the canonical map $\iota_H : H \to H''$ to the map

$$\Phi_H : H \to H', \quad \Phi_H(y) := \langle\cdot, y\rangle.$$

Problem 8 (bidual operator, canonical map, 3pt). Let X, Y be normed spaces and $T \in B(X, Y)$. Prove that

$$T'' \circ \iota_X = \iota_Y \circ T.$$

Problem 9 (adjoint of left-shift, 3 pt). Find the adjoint operator for

$$L : \ell^2 \to \ell^2, \quad Lx := (x^2, x^3, \ldots).$$

Remark: You may use that L is linear and bounded. You may also use the characterization of the adjoint operator that was proved in an assignment.

Problem 10 (spectrum, 7 pt). Let

$$T : X := C([0,1], \mathbb{C}) \to X, \quad (Tx)(t) := \int_0^t x(s)ds.$$

Show that the spectrum of T equals $\{0\}$.

Hint: Compute the spectral radius of T.

Problem 11 (continuous functions and ℓ^2, 15 pt). Let $A \subseteq [0,1]$ be a closed subset different from $[0,1]$. We define

$$X := \{x \in C[0,1] \mid x(t) = 0, \forall t \in A\}, \quad \|\cdot\| : X \to \mathbb{R}, \|x\| := \sqrt{\int_0^1 |x(t)|^2 dt}.$$

Prove that there exists a linear isometry $T : X \to \ell^2$, whose image is dense.

Remark: In this exercise you may use any exercise from the assignments (and any result from the lecture and the book by Rynne and Youngson).