F. Ziltener

blok 2, 2016/ 2017
February 1, 2017

Functionaalanalyse, WISB315

Tentamen

Family name: ____________________ Given name: ____________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Rynne and Youngson, without proving it.

If an exam problem was (part of) a result X in the lecture or in the book then you need to reprove the statement here. Unless otherwise stated, you may use any result that was used in the proof of X without proving it.

Unless otherwise stated, you may use without proof:

- A given map is linear (if this is indeed the case).
- $\big(C([0,1],\mathbb{R}), \|\cdot\|_\infty\big)$ is a Banach space.
- Under suitable hypotheses a norm on a vector space X induces a norm on the quotient X/Y, where Y is a linear subspace of X.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

27 points will suffice for a passing grade 6.

Good luck!

1	2	3	4	5	6	7	8	9	$\sum$
/8	/5	/5	/7	/10	/7	/4	/10	/9	/65

Problem 1 (closed linear subspace, 8 pt)**.** Let $\mathbb{K} = \mathbb{R}$. Prove that

$$c := \left\{ x \in \mathbb{R}^{\mathbb{N}} \,\middle|\, x^i \text{ converges as } i \to \infty \right\} \tag{1}$$

is a linear subspace and a closed subset of $\ell^\infty = \ell^\infty(\mathbb{N})$ (with respect to the supremum norm $\|\cdot\|_\infty$).

Problem 2 (norm of a product, 5 pt)**.** Let $\mathbb{K} = \mathbb{R}$, $p, q, r \in [1, \infty)$ be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

$x \in \ell^p = \ell^p(\mathbb{N})$, and $y \in \ell^q$. We define

$$xy : \mathbb{N} \to \mathbb{R}, \quad (xy)^i = (xy)(i) := x^i y^i.$$

Show that

$$xy \in \ell^r, \quad \|xy\|_r \le \|x\|_p \|y\|_q.$$

Problem 3 (separability, 5 pt)**.** Is $(\ell^\infty, \|\cdot\|_\infty)$ separable?

Problem 4 (integral equation, 7 pt)**.** Let $y \in C([0,1], \mathbb{R})$ and $k \in C([0,1]^2, \mathbb{R})$ be such that

$$\|k\|_\infty < 1.$$

Show that there exists a unique solution $x \in C([0,1], \mathbb{R})$ of the equation

$$x(t) - \int_0^1 k(t,s) x(s)\, ds = y(t), \quad \forall t \in [0,1]. \tag{2}$$

Remark: You may use the fact that the function $[0,1] \ni t \mapsto \int_0^1 k(t,s)x(s)\, ds \in \mathbb{R}$ is continuous.

Problem 5 (functional on ℓ^∞, 10 pt)**.** Let $\mathbb{K} = \mathbb{R}$. Prove that there exists a nonzero bounded linear map from ℓ^∞ to $\mathbb{R}$ that vanishes on c (as defined in (1)).

Remark: This follows from a corollary in the lecture/ in the book by Rynne and Youngson. You need to reprove all parts of the corollary that you use.

Problem 6 (dual of a Hilbert space, 7 pt)**.** Let H be a Hilbert space. Show that there exists a complete inner product on the dual space H' that induces the operator norm.

Problem 7 (criterion for boundedness of a set, 4 pt)**.** Let X be a normed vector space and $S \subseteq X$. Assume that for every $x' \in X'$ the set $x'(S) \subseteq \mathbb{K}$ is bounded. Prove that S is bounded.

Remark: Boundedness of S means that

$$\sup_{x \in S} \|x\| < \infty.$$

Hint: Use (without proof) a property of the canonical map ι_X and one of the three main theorems about operators on Banach spaces.

(More problems on the back.)

Problem 8 (self-adjoint operator on ℓ^2, 10 pt)**.** Let $k : \mathbb{N} \times \mathbb{N} \to \mathbb{K} := \mathbb{C}$ be a function, such that, denoting $k^i{}_j := k(i,j)$,

$$\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \left|k^i{}_j\right|^2 < \infty.$$

Show the following:

(i) Let $x \in \ell^2$ and $i \in \mathbb{N}$. The limit

$$y^i := \lim_{n\to\infty} \sum_{j=1}^{n} k^i{}_j x^j$$

exists.

We define

$$T_k(x) := (y^i)_{i\in\mathbb{N}}.$$

(ii) For every $x \in \ell^2$ we have

$$T_k(x) \in \ell^2.$$

(iii) The linear map $T_k : \ell^2 \to \ell^2$ is bounded. (You do not need to show that this map is linear.)

(iv) Assume that

$$k^i{}_j = \overline{k^j{}_i}, \quad \forall i,j \in \mathbb{N}.$$

Then T_k is self-adjoint.

Problem 9 (ℓ^∞ contains every separable space, 9 pt)**.** Show that for every separable normed vector space $(X, \|\cdot\|)$ there exists a linear isometry $i : X \to \ell^\infty$.

Hint: Use the Hahn-Banach Theorem.