F. Ziltener

blok 2, 2017/ 2018
January 31, 2018

Functionaalanalyse, WISB315

Tentamen

Family name: ______________________ Given name: ______________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, **not** with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are **not** allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Rynne and Youngson, without proving it.

If an exam problem was (part of) a result X in the lecture or in the book then you need to reprove the statement here. Unless otherwise stated, you may use any result that was used in the proof of X without proving it.

Unless otherwise stated, you may use without proof that a given map is linear (if this is indeed the case).

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

25 points will suffice for a passing grade 6.

Good luck!

1	2	3	4	5	6	7	8	9	$\sum$
/8	/9	/4	/3	/7	/10	/6	/6	/11	/64

Problem 1 (seminorm, 8 pt)**.** Let X be a real vector space and $\|\cdot\| : X \to [0,\infty)$ a map satisfying

$$\|ax\| = |a|\|x\|, \quad \forall a \in \mathbb{R}.$$

Show that $\|\cdot\|$ is a seminorm if and only if the closed unit ball

$$\{x \in X \,|\, \|x\| \leq 1\}$$

is convex.

Problem 2 (norm of product, 9 pt)**.** Let $\mathbb{K} = \mathbb{R}$, $p, q, r \in [1,\infty)$, such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

and $x \in \ell^p$, $y \in \ell^q$, $z \in \ell^r$. Consider the pointwise product

$$xyz : \mathbb{N} \to \mathbb{R}, \quad (xyz)^i := (xyz)(i) := x^i y^i z^i.$$

Show that $xyz \in \ell^1$ and

$$\|xyz\|_1 \leq \|x\|_p \|y\|_q \|z\|_r.$$

Hint: Use a similar statement involving only two sequences.

Problem 3 (Fourier coefficients, 4 pt)**.** Does there exist a function $f \in \mathcal{L}^2([0,1],\mathbb{C})$, whose n-th Fourier coefficient satisfies

$$\widehat{f}^n = \frac{1}{\sqrt{n}},$$

for every $n \in \mathbb{N}$?

Remark: Recall that

$$\mathcal{L}^2([0,1],\mathbb{C}) = \left\{f : [0,1] \to \mathbb{C} \,\middle|\, f \text{ measurable}, \int_{[0,1]} |u|^2 d\lambda < \infty\right\}$$

(integral with respect to the Lebesgue measure)

Problem 4 (equivalence of norms, 3 pt)**.** Let $\|\cdot\|$ and $|||\cdot|||$ be complete norms on a vector space X, such that there exists $C \in \mathbb{R}$ satisfying

$$\|x\| \leq C|||x|||, \quad \forall x \in X.$$

Show that $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

Hint: Use a result from the lecture.

Problem 5 (spectrum closed, 7 pt)**.** Let X be a complex Banach space and $T \in B(X)$. Prove that the spectrum $\sigma(T)$ is closed.

Remarks: This is a direct consequence of a lemma that was stated in the lecture and proved as an exercise. You need to reprove the statement here.

If you use that the set of invertible operators is open then you need to prove this here.

(Please turn over!)

Problem 6 (dual space of c_0, 10 pt)**.** Show that the map

$$\Phi_{c_0} : \ell^1 \to c_0', \quad \Phi_{c_0}(y)x := \sum_{i=1}^{\infty} x^i y^i,$$

is surjective.

Remarks: c_0' denotes the dual space of c_0. You may use without proof that Φ_{c_0} is well-defined and a linear isometry.

Hints: Given $y' \in c_0'$ guess how to define y satisfying $\Phi(y) = y'$, by evaluating y' on certain vectors. Approximate vectors in ℓ^1 and c_0 by vectors in

$$d = \left\{ x \in \mathbb{K}^{\mathbb{N}} \,\middle|\, \exists n \in \mathbb{N} : \, x^i = 0, \, \forall i \geq n \right\}.$$

Problem 7 (spectrum of self-adjoint operator, 6 pt)**.** Let H be a complex Hilbert space, $T : H \to H$ a self-adjoint bounded operator, and $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{pt}}(T)$, where

$$\sigma_{\mathrm{pt}}(T) := \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda \,\mathrm{id} - T \text{ not injective} \right\}.$$

Show that the image of $\lambda \,\mathrm{id} - T : H \to H$ is dense.

Problem 8 (dualization map surjective, 6 pt)**.** Let X, Y be reflexive normed spaces. Show that the dualization map

$$B(X, Y) \ni T \mapsto T' \in B(Y', X')$$

is surjective.

Problem 9 (boundedness of diagonal operator, 11 pt)**.** Let $\| \cdot \|$ be a norm on

$$d := \left\{ x = (x^i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \,\middle|\, \exists N \in \mathbb{N} : \, x^i = 0 \, \forall i \geq N \right\}$$

and $a_i \in \mathbb{R}$, for $i \in \mathbb{N}$. We define

$$T : d \to d, \quad T(x) := (a_i x^i)_{i \in \mathbb{N}}.$$

(i) Is the operator T bounded if

$$\sup_{i \in \mathbb{N}} |a_i| < \infty?$$

(ii) Is T bounded if for every $k \in \mathbb{N}$

$$\sup \left\{ \|Tx\| \,\middle|\, I \subseteq \mathbb{N}, \, x \in \overline{B}_1^{d, \|\cdot\|} : \, |I| = k, \, x^i = 0 \text{ if } i \in \mathbb{N} \setminus I \right\} < \infty?$$