F. Ziltener

blok 2, 2017/ 2018
April 20, 2018

Functionaalanalyse, WISB315

Hertentamen

Family name: ____________________ Given name: ____________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, **not** with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are **not** allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use any result (theorem, proposition, corollary or lemma) that was proved in the lecture or in the book by Rynne and Youngson, without proving it.

If an exam problem was (part of) a result X in the lecture or in the book then you need to reprove the statement here. Unless otherwise stated, you may use any result that was used in the proof of X without proving it.

Unless otherwise stated, you may use without proof that a given map is linear (if this is indeed the case).

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

23 points will suffice for a passing grade 6.

Good luck!

1	2	3	4	5	6	7	8	9	$\sum$
/4	/6	/5	/8	/7	/5	/6	/6	/12	/59

Problem 1 (Young's inequality, 4 pt)**.** Show that for every $a, b \in [0, \infty)$ and $1 < p < \infty$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \tag{1}$$

where

$$p' := \frac{p}{p-1}.$$

Hint: Consider $A := a^p$, $B := b^{p'}$, and take the logarithm.

Problem 2 (quotient seminorm, 6 pt)**.** Let $\big(X, \|\cdot\|\big)$ be a semi-normed vector space and $Y \subseteq X$ a linear subspace. Consider the map

$$\begin{aligned} \|\cdot\|^Y &: X/Y \to [0, \infty), \\ \|\overline{x}\|^Y &:= \inf\nolimits_{x \in \overline{x}} \|x\|. \end{aligned}$$

Prove that this map is a seminorm.

Remark: You do not need to prove that the quotient space X/Y is a vector space.

Problem 3 (polarization identity, 5 pt)**.** Let $\big(X, \langle \cdot, \cdot \rangle\big)$ be a Hermitian inner product space. (Hence $\mathbb{K} = \mathbb{C}$.) Show that

$$\langle x, y \rangle = \frac{1}{4}\Big(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\Big), \quad \forall x, y \in X.$$

Problem 4 (bounded linear functional on ℓ^p, 8 pt)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $p \in (1, \infty)$. We denote

$$\ell^p = \ell^p(\mathbb{N}) := \left\{ x = (x^i)_{i \in \mathbb{N}} \,\middle|\, \sum_{i \in \mathbb{N}} |x^i|^p < \infty \right\}, \quad p' := \frac{p}{p-1}.$$

Let $y \in \ell^{p'}$. Consider the map

$$T : \ell^p \to \mathbb{K}, \quad Tx := \sum_{i=1}^{\infty} x^i y^i.$$

(i) Show that T is well-defined, i.e., the series converges.

(ii) Show that T is linear and bounded.

(iii) Calculate its operator norm.

Problem 5 (dual of a Hilbert space, 7 pt)**.** Let H be a Hilbert space. Show that there exists a complete inner product on the dual space H' that induces the operator norm.

Problem 6 (pointwise limit of continuous operators, 5 pt)**.** Let X be a Banach space and Y a normed space. For $n \in \mathbb{N}$ let $T_n : X \to Y$ be a bounded linear operator. Assume that for every $x \in X$ the limit

$$T(x) := \lim_{n \to \infty} T_n(x)$$

exists. Show that the operator T is bounded. (It is linear, but you do not need to show this.)

(Please turn over!)

Problem 7 (canonical map, 6 pt)**.** Show that for every normed vector space X the canonical map $\iota_X : X \to X''$ is an isometry.

Remark: This was part of a proposition in the lecture, whose proof relied on a corollary. You need to reprove the relevant parts of the proposition and the corollary here. You do *not* need to prove that ι_X is well-defined, nor that it is linear.

Problem 8 (spectrum of multiplication operator on ℓ^p, 6 pt)**.** Let $p \in [1, \infty]$ and $y \in \ell^\infty$. We define

$$M_y : \ell^p \to \ell^p, \quad M_y x := yx = (y^i x^i)_{i \in \mathbb{N}}.$$

Prove that

$$\sigma_{\mathrm{pt}}(M_y) = \mathrm{im}(y) = \{ y^i \mid i \in \mathbb{N} \},$$
$$\sigma(M_y) = \overline{\mathrm{im}(y)},$$

where σ_{pt} denotes the point spectrum (= set of eigenvalues) and σ denotes the spectrum.

Remark: The map M_y is linear and bounded. You do not need to prove this.

Problem 9 (continuous functions and ℓ^2, 12 pt)**.** Let $A \subseteq [0, 1]$ be a closed subset different from $[0, 1]$. We define

$$X := \{ x \in C[0,1] \mid x(t) = 0, \forall t \in A \}, \quad \| \cdot \| : X \to \mathbb{R}, \, \|x\| := \sqrt{\int_0^1 |x(t)|^2 dt}.$$

Prove that there exists a linear isometry $T : X \to \ell^2$, whose image is dense.

Remark: In this exercise you may use any exercise from the assignments (and any result from the lecture and the book by Rynne and Youngson).