FINAL EXAM 'INLEIDING IN DE GETALTHEORIE'

Thursday, 10th January 2019, 14:00 pm - 17:00 pm

Question 1 (4 points)

a) Find the continued fraction expansion to $\sqrt{7}$.

b) What number has the continued fraction expansion

$$\langle 5, \overline{1, 1, 1, 10} \rangle \quad ?$$

Question 2 (4 points)

a) Find all integer solutions to the following system of congruences (i.e. integers x that simultaneously solve all of the following congruences):

$$x \equiv 3 \bmod 5$$
$$x \equiv 6 \bmod 11$$
$$x \equiv 7 \bmod 91.$$

b) Does the congruence

$$x^2 - 3x + 7 \equiv 0 \bmod 66$$

have a solution?

Question 3 (4 points)

Find the last digit of 7^{139} and 13^{2018}.

Question 4 (4 points)

a) Let $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. Show that

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \bmod mn.$$

Here $\phi(n)$ denotes Euler's phi-function.
b) Let $m > 4$ and assume that m is not prime. Show that

$$(m-1)! \not\equiv -1 \bmod m.$$

(in contrast to prime numbers.)

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Question 5 (4 points)

Consider the equation

$$x_1^3 + 2x_2^3 + 7(x_3^3 + 2x_4^3) + 7^2(x_5^3 + 2x_6^3) = 0$$

in the six variables $x_1, \ldots, x_6$. Show that the only solution over the integers is given by $x_1 = x_2 = \ldots = x_6 = 0$. Hint: work modulo 7.

Question 6 (4 points)

Assume that the *abc*-conjecture holds. Show that there are only finitely many solutions $a, b, c, d \in \mathbb{N}$ to the equation

$$a^{10} + b^{13} = c^8 d^9,$$

which satisfy $\gcd(a, b, cd) = 1$. Reminder: the natural numbers $\mathbb{N}$ do not contain 0 in the way that we defined it in the course.

Note: A simple non-programmable calculator is allowed for the exam.