FINAL EXAM 'INLEIDING IN DE GETALTHEORIE'

Thursday, 8th November 2018, 13.30 pm - 16.30 pm

Question 1 (4 points)

a) Find the continued fraction expansion to $\sqrt{41}$.
b) What number has the continued fraction expansion

$$\langle 4, \overline{1, 3, 1, 8} \rangle \quad ?$$

Question 2 (4 points)

a) Find all integer solutions to the following system of congruences (i.e. integers x that simultaneously solve all of the following congruences):

$$x \equiv 3 \bmod 6$$
$$x \equiv 6 \bmod 7$$
$$x \equiv 7 \bmod 143.$$

b) Does the congruence

$$x^2 - 2x + 3 \equiv 0 \bmod 105$$

have a solution?

Question 3 (4 points)
For a natural number m let $\phi(m)$ be Euler's phi-function, i.e. the number of invertible residue classes modulo m.

a) For what $n \in \mathbb{N}$ do we have $\phi(n) = 48$?

b) Compute the last digit of 3^{400}.

Question 4 (4 points)

Show that

$$x \equiv a \bmod m \quad \text{and} \quad x \equiv b \bmod n$$

have a common solution if and only if $\gcd(m, n) \mid b - a$, and in this case the solution is unique modulo the least common multiple of m and n.

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Question 5 (4 points)

Let $a, b, c, d \in \mathbb{Z}$ and $a \equiv d \equiv 4 \bmod 9$. Assume that the equation

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = z^3$$

has a nontrivial integer solution in x, y, z (i.e. a solution where not all of x, y, z are equal to zero). Show that in this case it also has an integer solution with $3 \nmid xy$.

Question 6 (4 points)

Assume that the abc-conjecture holds. Show that there are only finitely many solutions $a, b, c, d, e, f \in \mathbb{N}$ to the equation

$$a^8b^9 + c^8d^9 = e^8f^9,$$

which satisfy $\gcd(ab, cd, ef) = 1$. Reminder: the natural numbers $\mathbb{N}$ do not contain 0 in the way that we defined it in the course.

Note: A simple non-programmable calculator is allowed for the exam.