# Exam Foundations of Mathematics A with solutions

november 6, 2008, 14.00-17.00

This exam consists of 5 exercises; see also the back of this sheet.
Advice: first do those exercises you can do right away; then start thinking about the others. Good luck!

**Exercise 1**. Determine which of the following sets are countable or uncountable. Give a short explanation.

a) $\{x \in \mathbb{R} \mid \sin x \in \mathbb{Q}\}$

b) $\{f \in \{0,1\}^{\mathbb{N}} \mid \exists k \forall n \geq k f(n) = 0\}$

c) $\{A \subseteq \mathbb{N} \mid A \text{ is infinite}\}$

Solution: a) Since the finction sin is injective on each interval $[(n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$ and $\mathbb{Q}$ is countable, there are in each such interval only countably many $x$ with $\sin(x) \in \mathbb{Q}$; and $\mathbb{R}$ is a countable union of these intervals. So the whole set is a countable union of countable sets; hence countable.
b): Let $\{0,1\}^*$ be the set of finite 01-sequences. Then $\{0,1\}^*$ is countable, and there is a surjective function from $\{0,1\}^*$ to the set in the exercise (send a finite sequence $\sigma$ to the function which starts with $\sigma$ and has zeroes forever after); hence this set is countable too.
c): This set is equal to $\mathcal{P}(\mathbb{N}) - \mathcal{P}_{\text{fin}}(\mathbb{N})$. Now $\mathcal{P}(\mathbb{N})$ is uncountable and $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is countable, so the set in the exercise is uncountable.

**Exercise 2**. Let $X$ be a set, $L$ a well order, and $f : L \to X$ a surjective function. We define the following relation on $X$: $x < y$ holds if and only if for every $l \in L$ such that $f(l) = y$, there is a $k < l$ such that $f(k) = x$.

Prove, that this relation gives a well order on $X$.

Solution: define $s : X \to L$ by: $s(x)$ is the least $l \in L$ such that $f(l) = x$ (this is a good definition since $f$ is surjective and $L$ is a well-order). Now it is easy to see that for the given relation $<$ on $X$ we have: $x < y$ in $X$ precisely when $s(x) < s(y)$ in $L$. So $(X, <)$ is isomorphic to a subset of the well-order $L$ (with the order from $L$). Because every subset of a well-order is a well-order, $(X, <)$ is a well-order.

More directly, one can say: if $A \subseteq X$ is a nonempty subset, then because $f$ is surjective the subset $f^{-1}(A) = \{l \in L \mid f(l) \in A\}$ is nonempty and has therefore a least element $l_A$. You deduce easily that $f(l_A)$ is the least element of $A$ for the relation $<$ on $X$. So every nonempty subset has a least element, hence $(X, <)$ is a well-order.

**Exercise 3**. Suppose $A$ is a subset of $\mathbb{R}$. A real number $\xi$ is said to be *algebraic* over $A$, if there is a polynomial $P(X) = a_0 + a_1X + \cdots + a_nX^n$, with coefficients $a_0, \ldots, a_n$ from $A$, such that $P(\xi) = 0$. In this exercise you may use the known fact, that the number $e$ is *not* algebraic over $\mathbb{Q}$.

Prove that there is a subset $A$ of $\mathbb{R}$ with the following properties:

i) $e$ is not algebraic over $A$;

ii) every real number $\xi$ can be written as a quotient $\frac{P(e)}{Q(e)}$, where $P(X)$ and $Q(X)$ are polynomials with coefficients from $A$.

[Hint: apply Zorn's Lemma to the poset of those subsets $A \subset \mathbb{R}$ that satisfy: $0 \in A$, $1 \in A$ and $e$ is not algebraic over A]

Solution: the hint was a bit miserly; in fact, it was better to consider the poset $P$ of those subsets $A$ of $\mathbb{R}$ satisfying: a) $0, 1 \in A$ b) if $x \in A$ then $-x \in A$ c) $e$ is not algebraic over $A$. Suppose $\mathcal{C}$ is a chain in $P$; consider $\bigcup \mathcal{C}$. Clearly, $\bigcup \mathcal{C}$ satisfies a) and b); and if $e$ were algebraic over $\bigcup \mathcal{C}$ there would be a polynomial $P$ with coefficients in $\bigcup \mathcal{C}$ such that $P(e) = 0$; but every polynomial has only finitely many coefficients, so in fact there would already be a $C \in \mathcal{C}$ which contained all coefficients; then $e$ would be algebraic over $C$ which contradicts that $C \in P$. We conclude: if $\mathcal{C}$ is a chain in $P$ then $\bigcup \mathcal{C}$ is in $P$. So, $P$ satisfies the conditions of Zorn's Lemma and has a maximal element $A$. We prove b) for $A$:

If $\xi \in A$ then $\xi = \frac{\xi}{1}$, a quotient of constant polynomials with coefficients in $A$. If $\xi \notin A$ then by maximality of $A$, $A \cup \{\xi, -\xi\}$ is not a member of $P$ although it satisfies a) and b). Therefore, $e$ is algebraic over $A \cup \{\xi, -\xi\}$; let $P(e) = 0$ with coefficients in $A \cup \{\xi, -\xi\}$. Not all coefficients are in $A$ because

$e$ is not algebraic over $A$; and not all coefficients are $\pm\xi$ because that would imply $\xi = 0$ (contradicting that $\xi \notin A$), or $e$ algebraic over $A$, a contradiction in both cases. So $P(X)$ can be written as $Q(X) + \xi R(X)$, where $Q$ and $R$ are polynomials with coefficients in $A$. The relation $P(e) = 0$ can now be rewritten to $\xi = \frac{-Q(e)}{R(e)}$, and by condition c) also $-Q$ is a polynomial with coefficients in $A$, so this is of the desired form.

**Exercise 4**. In this exercise we consider the language $L_{\text{pos}}$ of posets: there is one binary relation symbol $\leq$.

For every natural number $n > 1$ we denote by $M_n$ the $L_{\text{pos}}$-structure which consists of all divisors of $n$, where we put $k \leq l$ precisely when $k$ is a divisor of $l$.

a) Give an $L_{\text{pos}}$-sentence which is true in $M_{32}$ but false in $M_{18}$;

b) The same for $M_{30}$ and $M_{24}$.

Give an explanation in words of what your sentences are intended to mean.

Solution: in the first case, you can see that $M_{32}$ is a linear order whereas $M_{18}$ is not; so you could take $\forall xy(x \leq y \vee y \leq x)$. Another possibility is to see that $M_{32}$ contains a chain of length 6 and $M_{18}$ does not; so you could take

$$\begin{aligned} &\exists x_1 \exists x_2 \cdots \exists x_6 (x_1 \leq x_2 \wedge x_2 \leq x_3 \wedge \cdots \wedge x_5 \leq x_6 \\ &\quad \wedge \neg(x_1 = x_2) \wedge \neg(x_2 = x_3) \wedge \cdots \wedge \neg(x_5 = x_6)) \end{aligned}$$

In the second case, you could write down in a similar way a sentence expressing "there is no chain of length 5", which is true in $M_{30}$ but false in $M_{24}$. Or, in $M_{30}$ "there are 3 parwise incomparable elements":

$$\begin{aligned} \exists x \exists y \exists z (\neg(x \leq y) \wedge \neg(y \leq x) \wedge \neg(x \leq z) \wedge \neg(z \leq x) \\ \wedge \neg(y \leq z) \wedge \neg(z \leq y)) \end{aligned}$$

**Exercise 5**. Again, we consider the language $L_{\text{pos}}$ of the previous exercise. Suppose $M$ is an infinite well order. Prove that there is a poset $M'$ with the following properties:

i) $M$ and $M'$ satisfy the same $L_{\text{pos}}$-sentences

ii) $M'$ is is not a well order.

[Hint: let $L^* = L_{\text{pos}} \cup C$, where $C = \{c_0, c_1, \ldots\}$ is a set of new constants. Define the following $L^*$-theory:

$$T \;=\; \{\phi \,|\, M \models \phi\} \,\cup\, \{c_{k+1} \leq c_k \wedge \neg(c_{k+1} = c_k) \,|\, k \in \mathbb{N}\}$$

Prove, using the Compactness Theorem, that $T$ has a model, and that every model of $T$ satisfies i) and ii).]

Solution: let's abbreviate $T_M$ for the set of $L_{\text{pos}}$-sentences true in $M$. If $T' \subset T$ is a finite subtheory then $T'$ is contained in $T_M \cup \{c_{k+1} < c_k \,|\, 0 \leq k \leq N\}$ for some $N \in \mathbb{N}$. Now because $M$ is infinite, it certainly contains a descending sequence of length $N+2$, hence interpretations for $c_0, \ldots, c_{N+1}$ in such a way that $T'$ is true in $M$. So $T'$ is consistent; hence by the Compactness Theorem $T$ is, and has a model $M'$. This model satisfies i): if $M \models \phi$ then $\phi \in T_M$ so $M' \models \phi$; if $M \not\models \phi$ then $M \models \neg\phi$ so $\neg\phi \in T_M$, whence $M' \not\models \phi$. Also, $M'$ is not a well-order because $M'$ contains an infinite descending chain.