**Exam: Representations of finite groups (WISB324)**

Wednesday June 29, 9.00-12.00 h.

- You are allowed to bring one piece of $A4$-paper, wich may contain formulas, theorems or whatever you want (written/printed on both sides of the paper).
- All exercise parts having a number $(\cdot)$ are worth 1 point, except for 1(f), 1(h), 2(e), 3(b) and 3(f) which are worth 2 points. Exercise 1(i) is a bonus exercise, which is worth 2 points.
- Do not only give answers, but also prove statements, for instance by refering to a theorem in the book.

*Good luck.*

1. Let $G$ be a non-commutative group of order 8.
(a) Show that there is no element of order 8.
(b) Show that there are elements of $G$ that have order 4.
(c) Show that $G$ has exactly 5 conjugacy classes and determine the degrees of the irreducible representations of $G$.

Now let $G = Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the Quaternion group, satisfying the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j\,.$$

(d) Determine all conjugacy classes of $Q$.
(e) Show that $\langle i \rangle$ (the group generated by $i$) is a normal subgroup of $Q$.
(f) Calculate the character table of $Q$.
(g) Determine the character of the regular representation of $Q$.
(h) Determine all normal subgroups of $Q$.
(i) (Bonus exercise) Find explicitly the matrices in $GL(n, \mathbb{C})$ for all elements of the irreducible representation of $Q$ for which $n$ is maximal.

2. Let $\mathbb{F} = \mathbb{C}$ and let $G$ be a group.
(a) Let $x \in G$, show that $C_x = \sum_{g \in x^G} g$ is in the center $Z(\mathbb{C}G)$ of the group algebra $\mathbb{C}G$.
(b) Show that $C_x = C_y$ if and only if $y \in x^G$.
(c) Let $G$ have $k$ conjugacy classes and let $x_1, x_2, \ldots, x_k$ be representatives of these different conjugacy classes. Show that $C_{x_1}, C_{x_2}, \cdots, C_{x_k}$ are linearly independent.
(d) Let $\chi_1, \chi_2, \ldots \chi_\ell$ be the collection of all irreducible characters of $G$, prove that $D_i = \sum_{g \in G} \chi_i(g^{-1}) g$ is in $Z(\mathbb{C}G)$.
(e) Prove that

$$\operatorname{span}(C_{x_1}, C_{x_2}, \ldots, C_{x_k}) = \operatorname{span}(D_1, D_2, \ldots, D_\ell)\,.$$

(f) Prove that the elements $D_i$ are also linearly independent.

3. Let $H \leq G$ and let $\chi$ be a character of $H$.
(a) Prove that $\chi \uparrow G(1) = [G : H]\chi(1)$.
(b) Which irreducible character of the Quaternion group $Q$ of exercise 1 is induced by a character of one of its subgroups?
(c) Let $H$ be in the center $Z(G)$ of $G$, prove that

$$\chi \uparrow G(g) = \begin{cases} [G : H]\chi(g) & if\ g \in H, \\ 0 & if\ g \notin H. \end{cases}$$

From now on let $G = D_{4n} = \langle a, b | a^{2n} = b^2 = 1,\ ab = ba^{-1} \rangle$.
(d) Determine the center $Z(D_{4n})$ of $D_{4n}$.
(e) Let $n \geq 2$, $H = Z(D_{4n})$ and $\chi$ be the non-trivial irreducible character of $H$, determine the values of $\chi \uparrow G(g)$ for $g \in D_{4n}$.
(f) The irreducible characters of $D_{4n}$( $n \geq 2$) have the following values on 1 and $a^n$:

- $(\psi(1), \psi(a^n)) = (1, 1)$,
- $(\psi(1), \psi(a^n)) = (1, -1)$,
- $(\psi(1), \psi(a^n)) = (2, 2)$,
- $(\psi(1), \psi(a^n)) = (2, -2)$.

Determine in all 4 cases the multiplicity of $\psi$ in $\chi \uparrow G$.