Exam: Representations of finite groups (WISB324)

Wednesday June 28 2017, 9.00-12.00 h.

- You are allowed to bring one piece of $A4$-paper, wich may contain formulas, theorems or whatever you want (written/printed on both sides of the paper).
- All exercise parts having a number $(\cdot)$ are worth 1 point, except when otherwise stated. With 20 points you have a 10 as grade for this exam. There is one bonus exercise of 2 points.
- Do not only give answers, but also prove statements, for instance by referring to a theorem in the book.

Good luck.

1. Consider the group D_{2n} for n odd and $n > 2$ with generators a and b and relations $a^n = 1$, $b^2 = 1$ and $bab = a^{n-1}$. We define a representation ρ on the vector space of complex polynomials in n variables $\mathbb{C}[x_1, x_2, \cdots x_n]$ by defining that $\rho(a)(x_j) = x_{j+1 (\mod n)}$ and $\rho(b)(x_j) = x_{n-j+1}$. We extend this to monomials as follows:
$$\rho(g)(x_{i_1}x_{i_2}\cdots x_{i_k}) = \rho(g)(x_{i_1})\rho(g)(x_{i_2})\cdots\rho(g)(x_{i_k}).$$
(a) Show that this indeed defines a representation of D_{2n}.
(b) Show that
$$V_m = \{p \in \mathbb{C}[x_1, x_2, \cdots x_n] |\, p \text{ homogeneous of degree } m\}$$
is a $\mathbb{C}D_{2n}$-module.
(c) Show that V_m is not irreducible.
(d) (Bonus exercise, 2 points) Decompose V_1 into a direct sum of irreducible $\mathbb{C}D_{2n}$-submodules.

2. Let G be a group ψ a non-trivial linear character and χ the only irreducible character of degree $n > 1$.
(a) Prove that $\psi\chi$ is also an irreducible character and that $\psi\chi = \chi$.
(b) Prove that $\chi(g) = 0$ if $\psi(g) \neq 1$.

3. Let G be a group with generators a and b and relations $a^7 = 1$, $b^6 = 1$ and $b^{-1}ab = a^3$. The subgroup generated by a is denoted by H.
(a) Show that H is a normal subgroup of G and that G/H is abelian.
(b) List all conjugacy classes of G by giving one element in each conjugacy class.
(c) Determine the degrees of the irreducible characters of G.
(d) (2 points) Determine the complete character table of G.
(e) Determine all normal subgroups of G.
(f) Let χ be a non-trivial character of the subgroup H. Compute the induced character $\chi \uparrow G$ and show that this is an irreducible character.

4. Let G be a finite group with character χ. We call χ real if $\chi(g) \in \mathbb{R}$ for all $g \in G$.
(a) Prove that all characters of G are real if and only if all irreducible characters of G are real.
Let $p > 2$ be a prime number and assume that C_p is a normal subgroup of G such that $|G| = mp$ and $\gcd(m, p-1) = 1$.
(b) Prove $|\mathrm{Aut}\, C_p| = p - 1$.
Let $a \in G$ and define the automorphism $\rho_a : C_p \to C_p$ by $\rho_a(x) = axa^{-1}$ for $x \in C_p$.
(c) Show that $\rho_a \rho_b = \rho_{ab}$ and prove that $\rho_a^m = 1$.
(d) Prove that $\rho_a = 1$.
(e) Let ϕ be a character of C_p. Prove that the induced character $\phi \uparrow G$ staisfies

$$\phi \uparrow G(x) = \begin{cases} m\phi(x) & \text{if } x \in C_p, \\ 0 & \text{if } x \notin C_p. \end{cases}$$

(f) Prove that not all characters of G are real.

5. (2 points) Let G be a group and H a subgroup. Let χ be a character of G and ψ a character of H. Prove Frobenius Reciprocity Theorem by elementary calculations, using the definitions of or formulas for the induced and resticted characters. Frobenius Reciprocity Theorem states that

$$\langle \psi, \chi \downarrow H \rangle_H = \langle \psi \uparrow G, \chi \rangle_G.$$