Exam: Representations of finite groups (WISB324)

Wednesday July 19 2017, 9.00-12.00 h.

- You are allowed to bring one piece of $A4$-paper, wich may contain formulas, theorems or whatever you want (written/printed on both sides of the paper).
- All exercise parts having a number $(\cdot)$ are worth 1 point, except when otherwise stated. With 20 points you have a 10 as grade for this exam. There are two bonus exercises of 1 point.
- Do not only give answers, but also prove statements, for instance by referring to a theorem in the book.

Good luck.

1. Let G be a finite group, V a $\mathbb{C}G$-module, $\langle \cdot, \cdot \rangle$ a complex inner product on V that is G-invariant, i.e., $\langle gv, gw \rangle = \langle v, w \rangle$ for all $v, w \in V$ and $g \in G$.
(a) Let $U \subset V$ be a $\mathbb{C}G$-submodule, show that $U^\perp$ is also a $\mathbb{C}G$-submodule and that $V = U \oplus U^\perp$.

From now on, let G be the symmetric group S_n and let $V = \mathbb{C}^n$ be the permutation module, i.e., let $e_1, e_2, \ldots e_n$ be a basis of V, the permutation representation is defined as follows:

$$\rho(\pi)(e_j) = e_{\pi(j)} \text{ for } \pi \in S_n.$$

(b) Show that the character χ_V of V is equal to

$$\chi_V(g) = |\text{fix}\, g|, \text{ where } \text{fix}\, g = \{e_j | \, \rho(g)(e_j) = e_j\}.$$

(c) Find a one-dimensional irreducible submodule U of V and calculate its character χ_U.
(d) Show that the standard inner product on V, defined by $\langle e_i, e_j \rangle = \delta_{ij}$ is S_n-invariant and find $U^\perp$.
(e) Show that $\psi(g) = \text{fix}\, g - 1$ is also a character of S_n.

From now on let $n = 4$.
(f) Give a representative of all conjugacy classes of S_4, calculate the corresponding values for χ_U and ψ and show that ψ is irreducible.
(g) χ_U is a linear character. Find another linear character of S_4 and call this ϕ and show that $\phi\psi$ is also irreducible.
(h) Determine the character table of S_4.
(i) Determine the symmetric and alternating characters, χ_S and χ_A for all the irreducible characters in the character table of S_4. Show which ones are irreducible.
(j) (Bonus exercise, 1 point) Express all symmetric and alternating characters in terms of the irreducible ones.
(k) (Bonus exercise, 1 point) Give for all irreducible $\mathbb{C}S_4$-modules W the decomposition of $W \otimes W$ as direct sum of irreducible modules.

2. Let ρ be a representation of the group G over $\mathbb{C}$.
(a) Show that $\delta : g \mapsto \det(\rho(g))$ for all $g \in G$ is a linear character of G.
(b) Prove that $G/\operatorname{Ker}\delta$ is abelian.
(c) Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

3. Let G be the group generated by a and b and relations $a^7 = b^3 = 1$ and $b^{-1}ab = a^2$. The subgroup generated by a is called H.
(a) Show that H is a normal subgroup of G and that G/H is abelian.
(b) Show that G has 5 conjugacy classes and give a representative of each conjugacy class.
(c) Determine the degrees of the irreducible representations.
(d) Give all linear characters of G.
(e) (2 points) Give the complete character table of G.
(f) Determine all normal subgroups of G.
(g) Let K be the subgroup generated by b, determine the non-trivial irreducible characters of K and the corresponding induced characters of G.