EXAM

(27^{th} June 2018, 9:00 – 11:00)

- You are allowed to use one A4-sheet (recto-verso) of handwritten notes. By contrast, you are not allowed to use any book or electronic devices during the exam.
- Every question part is worth 1 point except where indicated otherwise. If you obtain 20 points, you are guaranteed a grade 10.
- There are two bonus questions at the end. You can use them to gain additional points but answering them is not necessary to obtain the maximum grade.
- Do not just give answers but prove your statements, e.g. by referring to a theorem in the course.

Question 1. The G be a finite group.

(a) Give the definition of the centre $Z(G)$ and prove that it cannot have prime index.

(b) Prove that if X is a G-set and x, $y \in X$ two elements in the same orbit, then their stabilisers G_x, $G_y \leqslant G$ are conjugate.

Question 2. Let G be a group of order 15. Show that every irreducible character of G has dimension 1 and deduce that G is abelian.

Question 3. Let k be a finite field, $G := \mathrm{SL}_2(k)$ and $H \leqslant G$ the subgroup of those $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G$ with $c = 0$. Further, let $\varphi\colon k^\times \to \mathbb{C}^\times$ be a group homomorphism (where $k^\times$ and $\mathbb{C}^\times$ are groups under multiplication) and χ the linear character of H defined by

$$\chi\colon H \to \mathbb{C}^\times, \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto \varphi(a).$$

(a) (2 points) For $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G$, determine the groups $H_g := gHg^{-1} \cap H \leqslant H$.
Hint: Distinguish the cases $g \in H$ and for $g \notin H$, the cases $a = 0$ and $a \neq 0$.

(b) (2 points) Also determine the characters ${}^g\chi\colon H_g \to \mathbb{C}^\times$, $x \mapsto \chi(g^{-1}xg)$.

(c) Show that $\langle \chi{\downarrow}_{H_g}, {}^g\chi \rangle_{H_g} = 0$ iff $\langle \varphi, \bar{\varphi} \rangle_{k^\times} = 0$.

(d) Conclude that $\chi{\uparrow}^G$ is irreducible iff $\varphi^2\colon x \mapsto \varphi(x)^2$ is not constantly 1.

Question 4. Let $\mathbb{F}_3 = \{0, \pm 1\}$ be the field with 3 elements, $G := \mathrm{SL}_2(\mathbb{F}_3)$ (which is a group under matrix multiplication) and write

$$A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in G.$$

The conjugacy classes of G can be calculated by hand and we list them at the end of this question.

Date: 27^{th} June, 2018

(a) Without referring to the table below, show that $|G| = 24$.

(b) Determine the centralisers of A and B.

(c) Without referring to the table below, why are A, $-A$ and B pairwise not conjugated?

(d) Recall that $Q_8 = \langle i, j \mid i^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1}\rangle = \{\pm 1, \pm i, \pm j, \pm k\}$. Show that

$$\varphi\colon Q_8 \to G,\ i \mapsto B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ j \mapsto C := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is a well-defined morphism of groups and calculate its image.

(e) Conclude that $[G, G] = \mathrm{Im}(\varphi)$ and that $G/[G, G] \cong C_3$.

(f) Determine the degrees of the irreducible characters of G.

(g) Calculate the linear characters of G.

(h) (2 points) Consider the 1-dimensional projective space (with coefficients in $\mathbb{F}_3$)

$$X := P^1\mathbb{F}_3 := ({\mathbb{F}_3}^2 \setminus \{0\})/\{\pm 1\} = \{(1:0), (1:1), (1:-1), (0:1)\}$$

consisting of all non-zero vectors in ${\mathbb{F}_3}^2$ up to non-zero scalar multiple (so for example $(1:-1) = (-1:1)$). Show that G acts on X by matrix multiplication and determine the associated reduced permutation character $\chi_X - 1$. Is it irreducible?

(i) (2 points) One can show that G has exactly three real irreducible characters. With this information, complete the character table of G.

(j) (Bonus) Let $H \leqslant G$ be as in Question 3 (for the specific case $G = \mathrm{SL}_2(\mathbb{F}_3)$ at hand). Show that H is cyclic of order 6.

(k) (Bonus, 2 points) Picking an isomorphism $H \cong C_6 = \langle r \mid r^6 = e\rangle$ and letting ψ be the linear character of C_6 given by $\psi(r) := -1$, determine the induced character $\psi\!\uparrow^G$ and decompose it as a sum of irreducible ones.

Class	Size	Elements
E^G	1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$(-E)^G$	1	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
B^G	6	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$
A^G	4	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$
$(-A)^G$	4	$\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$
$(A^2)^G$	4	$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$
$(-A^2)^G$	4	$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$