Exam Representations of Finite Groups, WISB324

June 25, 2019, 17:00-20:00

- Write your name on every sheet.
- The book may be consulted.
- In each item you can use the results from previous items, even if you have not solved them.
- Motivate your solutions!
- Success!

1. Let G be the finite group given by
$$G = \left\langle a, b, c | a^3 = b^3 = c^3 = e, ab = ba, ac = ca, c^{-1}bc = ab \right\rangle .$$
It has 27 elements and 11 conjugation classes. In the following we compute the irreducible characters of G without computing the conjugation classes.
   (a) (1/2 pt) Determine the dimensions of the irreducible representations of G.
   (b) (1 pt) Determine the one-dimensional representations of G.
   (c) (1/2 pt) Show that $\{e\}, \{a\}, \{a^2\}$ are conjugation classes of G.
   (d) (1/2 pt) Show that $\chi(g) = 0$ for every $g \notin \{e, a, a^2\}$ and every irreducible character χ with $\chi(e) > 1$.
   (e) (1/2 pt) Show that $\chi(a^2) = \overline{\chi(a)}$ for every character χ.
   (f) (1/2 pt) Show that there is an irreducible character such that $\chi(a) \notin \mathbb{R}$. Define $\alpha = \chi(a)$ for this character.
   (g) (1 pt) Determine the possible values of α.

2. Consider the vector space of bilinear polynomials in $x_1, x_2, x_3, y_1, y_2, y_3$ given by
$$V = \left\{ \sum_{i,j=1}^{3} \lambda_{ij} x_i y_j \,\middle|\, \lambda_{ij} \in \mathbb{C} \right\} .$$
We give V a $\mathbb{C}S_3$-module structure by letting every $\sigma \in S_3$ action as $\sigma : x_i y_j \mapsto x_{\sigma(i)} y_{\sigma(j)}$.
   (a) (1/2 pt) Write down the character table of S_3. Briefly motivate your answer.
   (b) (1 pt) Determine the character of the $\mathbb{C}S_3$-module V and write it as sum of irreducible characters of S_3.

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(c) (1 pt) Write down generators of the subspaces of V that correspond to one-dimensional $\mathbb{C}S_3$ submodules of V.

(d) (1/2 pt) Show that the $\mathbb{C}S_3$-module V is isomorphic to $W \otimes W$, where W is the $\mathbb{C}S_3$-module given by the permutation representation $\sigma : \mathbf{e}_i \mapsto \mathbf{e}_{\sigma(i)}$ for all $\sigma \in S_3$ and $i = 1, 2, 3$.

3. Let χ be a character of a finite group G.

(a) (1 pt) Show that if $\chi(g) = 0$ for all $g \neq e$, then χ is a multiple of χ_{reg}, the character of the regular $\mathbb{C}G$-module.

(b) (1 pt) Suppose that $\chi(g) \in \mathbb{R}_{\geq 0}$ for all $g \in G$. Show that χ is either the trivial character, or reducible.

4. (1 bonus point) The regular representation of a finite group G consists of the vector space $\mathbb{C}G$ together with an action of G given by $\rho_1(g) : r \mapsto gr$ for all $g \in G, r \in \mathbb{C}G$. Denote this $\mathbb{C}G$-module by V_1. We define a second action of G by $\rho_2(g) : r \mapsto rg^{-1}$ for all $g \in G, r \in \mathbb{C}G$.

(a) (1/2) Show that $\mathbb{C}G$ with the action ρ_2 is a $\mathbb{C}G$-module. Denote it by V_2.

(b) (1/2) Show that V_1 and V_2 are isomorphic $\mathbb{C}G$-modules by exhibiting a $\mathbb{C}G$-isomorphism between them.