Retake Representations of finite groups

July 15, 2019

- Write your name on every sheet.
- The book may be consulted.
- In each item you can use the results from previous items, even if you have not solved them.
- Motivate your solutions!
- There are 11 pts to be earned. Success!

1. Let G be the group generated by three elements a, b, c subject to the relations $a^3 = b^3 = c^2 = e,\ ab = ba,\ cac = a^2,\ cbc = b^2$ (e is the neutral element in G). The group G has order 18 and each element can be written in the form $a^i b^j c^k$ with $i, j \in \{0, 1, 2\}$, $k \in \{0, 1\}$.

   (a) (1/2 pt) Determine the six conjugation classes of G.

   (b) (1 pt) Determine the one-dimensional representations of G.

   (c) (1/2 pt) Show that all other irreducible representations of G have dimension 2.

   (d) (1/2 pt) Let $\omega = e^{2\pi i/3}$ and define the matrices

   $$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad B = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Prove that the map $\rho : G \to GL(2, \mathbb{C})$ given by $\rho(a^i b^j c^k) = A^i B^j C^k$ for $0 \leq i, j \leq 2$ and $k = 0, 1$ is a two-dimensional representation of G. Compute the character of ρ.

   (e) (1/2 pt) Prove that ρ is irreducible.

   (f) (1 pt) Compute the character table of G (hint: use a variation of the construction in the previous item)

PTO/ZOZ

2. We are given the group A_5 of even permutations of 5 objects.

   (a) (1/2 pt) Give the conjugation classe of A_5.

   Let U be the 5-dimensional complex vector space of linear forms (homogeneous linear polynomials) in x_1, x_2, x_3, x_4, x_5. Define the representation $\pi : A_5 \to GL(U)$ by

$$\pi(\sigma) : L(x_1, x_2, x_3, x_4, x_5) \mapsto L(x_{\sigma(1)}, \ldots, x_{\sigma(5)})$$

   for every $L \in U$.

   Let V be the complex 10-dimensional vectorspace van of poynomials in x_1, x_2, x_3, x_4, x_5 spanned by $x_i x_j$ with $1 \leq i < j \leq 5$ (quadratic monomials with distinct indices) Define the representation $\rho : A_5 \to GL(V)$ by

$$\rho(\sigma) : Q(x_1, x_2, x_3, x_4, x_5) \mapsto Q(x_{\sigma(1)}, \ldots, x_{\sigma(5)})$$

   for all $Q \in V$.

   (b) (1/2 pt) Determine the character ψ of π and give it in a table.

   (c) (1/2 pt) Determine the character χ of ρ and give it in a table.

   (d) (1 pt) Show that V has a A_5-invariant subspace W of dimension 5 $\mathbb{C}A_5$-isomorphic to U. Do this by displaying a basis of W.

   (e) (1/2 pt) Show that $\chi - \psi$ is the character of an irreducible representation *without using the character table of* A_5.

3. In this problem G is a finite group and $|G|$ denotes the order of G. We fix an irreducible character χ of G and consider the element $X = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g$ in the group algebra $\mathbb{C}G$. We let U be a $\mathbb{C}G$-module and denote its character by ψ. Moreover we define the $\mathbb{C}$-linear map $\xi : U \to U$ by $\chi(v) = Xv$ for all $v \in U$.

   (a) (1/2 pt) Show that the trace of the $\mathbb{C}$-linear map ξ equals $\langle \psi, \chi \rangle$.

   (b) (1/2 pt) Prove that $h^{-1} X h = X$ for every $h \in G$.

   (c) (1/2 pt) Prove that ξ is a $\mathbb{C}G$-homomorphism.

   (d) (3/2 pt) For this sub-item assume that U is an irreducible $\mathbb{C}G$-module.

      i. Prove that there is a $\lambda \in \mathbb{C}$ such that $\xi(v) = \lambda v$ for all $v \in U$.

      ii. Prove that $\lambda = 0$ if $\psi \neq \chi$.

      iii. Compute λ if $\psi = \chi$.

   (e) (1/2 pt) Prove that $\xi(\xi(v)) = \frac{1}{\chi(1)} \xi(v)$ for every $v \in U$.

   (f) (1/2 pt) Prove that $X^2 = \frac{1}{\chi(1)} X$ holds in the group algebra $\mathbb{C}G$.