Final Exam
Representation of Finite Groups
24.06.2020, 9:00-12:00

Important

You have to upload your solution until 12:30.
Students with extra time have 30 minutes more. If that is true for you, you have to upload until 13:00.

You have to add the following declaration to your solution:

Hierbij verklaar ik dat ik de uitwerkingen van dit tentamen zelf heb gemaakt, zonder hulp van andere personen of van het internet.

There are 43 points to earn in the exam. 3 of those are bonus points. That means you receive the maximal grade if you get 40 or more points. You can use the statement of previous parts of a question even if you were not able to prove them.

Question 1

Let $G = \langle a, b : a^5 = b^4 = 1 \ , \ b^{-1}ab = a^3 \rangle$. It can be shown that G has 20 elements of the form $a^k b^i$ with $k \in \{0, \dots, 4\}$ and $i \in \{0, \dots 3\}$. Let H be the subgroup of G generated by a.

a) Show that H is a normal subgroup and that G/H is abelian. (2 points)

b) Determine the five conjugacy classes of G. (3 points)

c) Find all linear characters of G. (2 points)

d) Find the complete character table of G. (2 points)

e) Find all normal subgroups of G. (2 point)

Question 2

Let G be a finite group. In this question we will prove the following result:

Theorem 1. *The sum of all elements in any fixed row of the character table of G is a non negative integer.*

For this, we consider the action of G on the group algebra by conjugation, i.e. for $g \in G$, $h \in \mathbb{C}[G]$ we define

$$g * h = ghg^{-1}.$$

a) Show that the above multiplication makes $\mathbb{C}[G]$ into a $\mathbb{C}G$ module. (2 points)

b) Let χ_C be the character of this module. Show that $\chi_C(g) = |C_G(g)|$. (= The number of elements of the centralizer of g.) (2 points)

c) Use b) to show that the sum of elements in a row of the character table associated with a given character χ can be expressed as $\langle \chi, \chi_C \rangle$. (2 points)

d) Using c), complete the proof of the theorem. (2 points)

Question 3

Let $D_8 = \langle a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ and consider

$$V = \{ \sum_{i,j=0}^{3} \lambda_{i,j} x_i y_j : \lambda_{i,j} \in \mathbb{C} \}.$$

This is a 16 dimensional subspace of the vector space of polynomials, with basis $\{x_i y_j : i, j \in \{0, 1, 2, 3\}\}$. We define a group action on V by describing the multiplication of generators of D_8 with this basis. We set

$$a(x_i y_j) = x_{i+1(4)} y_{j-1(4)},$$

where we denote by (4) that the index is calculated modulo 4, so $3 + 1 = 0(4)$ and $0 - 1 = 3(4)$. Further we set

$$b(x_i y_j) = x_j y_i.$$

Recall that you can find the character table for D_8 at 16.3(3) in the book.

a) Show that the above multiplication makes V to a D_8 module. (2 points)

b) Calculate the character of V, say χ. (2 points)

c) Decompose χ into irreducible characters. (3 points)

d) Find a basis for each of the submodules of V that is isomorphic to the the trivial module. (Hint: Use 14.26) (3 points)

e) Find a basis for all irreducible two dimensional submodules of V. (Hint: Use again 14.26 and then group the found vectors suitably into pairs.) (3 points)

Question 4

Let G be a finite group and $n \in \mathbb{N}$. Define for $g \in G$ the function

$$r_n(g) = |\{h \in G : h^n = g\}|.$$

That is, $r_n(g)$ counts the number of ways g can be expressed as an n-th power.

a) Show that $r_n(g)$ is constant on conjugacy classes. (So it is a so-called class function.) (2 points)

b) Deduce from a) that there are $a_\chi \in \mathbb{C}$ such that $r_n(g) = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi(g)$, where the sum runs over all irreducible characters of G. (1 point)

c) Given an irreducible character χ of G, find an expression for a_χ. (3 points)

d) Let G be abelian and χ_1 denote the trivial character of G. Show that there exists a character of G, say ψ, such that $a_\chi = \langle \psi, \chi_1 \rangle$. (3 bonus points)

e) Conclude that, if G is abelian, $r_n(g)$ is a character of G. (2 points)