Geometry and Topology – Exam 3

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt)**.** For each of the lists below, decide which spaces are homotopy equivalent to each other

- $$(S^1 \times S^1)\backslash\{p\}, \qquad S^1 \vee S^1, \qquad \mathbb{K}\backslash\{p\}, \qquad S^2\backslash\{p_1, p_2, p_3\},$$
  where $\mathbb{K}$ denotes the Klein bottle.
- $$T^2 \# T^2, \qquad S^1 \times S^1 \times S^1 \times S^1, \qquad T^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2, \qquad \mathbb{K} \# \mathbb{K}.$$
- $$\mathbb{C}P^n, \qquad S^{2n}, \qquad \mathbb{R}P^{2n}.$$

Exercise 2 (1.0 pt)**.**

a) Let X be a CW complex with the property that $X = A \cup B$ where A, B and $A \cap B$ are contractible subcomplexes. Show that X is contractible.

b) Give an example of a CW complex X which can be decomposed as a union $X = A \cup B$ with both A and B contractible subcomplexes but for which X is not contractible.

Exercise 3 (1.0 pt)**.** Let $X = X_1 \cup \cdots \cup X_n$ be a subset of $\mathbb{R}^m$ such that each X_i is a convex set and $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j and k. Show that $\pi_1(X, x_0) = \{0\}$ for all $x_0 \in X$.

Exercise 4 (1.0 pt)**.** Show that if X is path-connected, locally path-connected and $\pi_1(X)$ is finite then every map $f : X \to S^1$ is null homotopic.

Exercise 5 (1.0 pt)**.** Given a map $f : S^{2n} \to S^{2n}$, show that there is a point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Conclude that every map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

Exercise 6 (2.0 pt)**.** Compute the homology groups of the following spaces.

a) The quotient space of S^2 obtained by identifying the north and south poles to a single point.

b) The quotient space of the disjoint union of two 2-tori obtained by identifying the circle $\{x_0\} \times S^1$ in one torus with the same circle in the other torus

Exercise 7 (2.0 pt)**.** Show that $H_i(X \times S^n) = H_i(X) \oplus H_{i-n}(X)$ for all i and n, where $H_i = 0$ for $i < 0$ by definition. Hint: show $H_i(X \times S^n) = H_i(X) \oplus H_i(X \times S^n; X \times \{x_0\})$ and $H_i(X \times S^n; X \times \{x_0\}) = H_{i-1}(X \times S^{n-1}; X \times \{x_0\})$.