Mathematisch Instituut, Faculteit Wiskunde en Informatica, UU.
In elektronische vorm beschikbaar gemaakt door de TBC van A–Eskwadraat.
Het college WISB342 werd in 2003 gegeven door Eduard Looijenga.

Uitwerking[1] Differentieerbare variëteiten (WISB342)
1 november

Recall that all maps and manifolds are assumed to be C^∞ unless the contrary is explicitly stated.

Opgave 1. Let $f : M \to N$ and $g : N \to P$ be maps between manifolds.

a) Prove that if f and g are submersions, then so is gf.

b) Prove that if f and g are immersions, then so is gf.

c) Prove that if f and g are embeddings, then so is gf.

Solution:

If $p \in M$, then $D_p(gf)$ is the composite of D_pf and $D_{f(p)}g$. So if the latter two are both surjective (rep. injective), then so is the third. This proves (a and (b. As to (c, recall that an embedding is the same thing as an immersion which is also a homeomorphism onto its image. So we need to prove that is f defines a homeomorphism of M onto the subspace $f(M)$ of N and g defines a homeomorphism of N onto the subspace $g(N)$ of P, then gf defines a homeomorphism of M onto the subspace $gf(M)$ of P. This is clear, for then g restricts to a homeomorphism of $f(M)$ onto the subspace $g(f(M))$ of P and hence gf defines a homeomorphism of M onto the subspace $g(f(M))$ of P.

Opgave 2.

a) Prove that the tangent bundle of S^3 is trivial.

b) Is the tangent bundle of P^3 trivial?

Solution:

The tangent bundle TS^n can be identified with the submanifold of $\mathbb{R}^{n+1} \times S^n$ of $(v, p) \in \mathbb{R}^{n+1} \times S^n$ with $v \perp p$. Regard S^3 as the unit sphere in the algebra $\mathbb{H} := \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ of quaternions. For every $p \in S^3$, (p, ip, jp, kp) is an orthonormal basis of $\mathbb{H}$, and so (ip, jp, kp) can be regarded as an orthonormal basis for the tangent space T_pS^3. Thus

$$(a, b, c; p) \in \mathbb{R}^3 \times S^3 \mapsto ((ai + bj + ck)p, p) \in TS^3$$

is an isomorphism of vector bundles.

The projective space P^n is obtained by dividing out by the involution $p \mapsto -p$. Its tangent bundle TP^n is obtained from $TS^n \subset \mathbb{R}^{n+1} \times S^n$ in similar fashion: identify (v, p) with $(-v, -p)$. In terms of the vector bundle isomorphism above this amounts to identifying $(a, b, c; p)$ with $(a, b, c; -p)$. The result is then $\mathbb{R}^3 \times P^3$. In other words,

$$(a, b, c; \pm p) \in \mathbb{R}^3 \times P^3 \mapsto \pm((ai + bj + ck)p, p) \in TP^3$$

is a vector bundle isomorphism. So the tangent bundle of P^3 is trivial.

Opgave 3. The Möbius strip M can be defined as follows: let $M_0 := (-\pi, \pi) \times (-1, 1)$ and $M_1 := (0, 2\pi) \times (-1, 1)$ and identify the open subset $U_0 := ((-\pi, \pi) - \{0\}) \times (-1, 1)$ of M_0 with the open subset $U_1 := ((0, 2\pi) - \{\pi\}) \times (-1, 1)$ of M_1 by means of the diffeomorphism $h : U_0 \to U_1$

$$h(x, y) = \begin{cases} (x + 2\pi, -y) \in U_1 & \text{in case } x \in (-\pi, 0); \\ (x, y) \in U_1 & \text{in case } x \in (0, \pi). \end{cases}$$

You may assume that M is a is a Hausdorff space and that the inverses of the maps $M_0 \to M$, $M_1 \to M$ define an atlas.

a) Prove that there is a vector field V on M whose restriction to M_0 resp. M_1 is given by $\partial/\partial x$.

b) Prove that V generates a flow $H : \mathbb{R} \times M \to M$ on M. Describe this flow in terms of the coordinates (x, y) on M_0 and M_1. Show that $H_{4\pi}$ is the identity map, but that $H_{2\pi}$ is not.

c) Explain why M is not orientable.

d) Let $N \subset M$ be the complement of the central circle (so where $y \neq 0$ on both M_0 and M_1). Prove that N is diffeomorphic to the open cylinder $S^1 \times (0, 1)$. Is N orientable?

Solution:

[1] Deze uitwerkingen zijn met de grootste zorg gemaakt. In geval van fouten kan de TBC niet verantwoordelijk worden gesteld, maar wordt zij wel graag op de hoogte gesteld: tbc@a-eskwadraat.nl

a) Denote the inverse of the $M_i \to M$ by κ_i. Then $\kappa_1\kappa_0^{-1} : U_0 \to U_1$ is described by the diffeomorphism h above. This diffeomorphism takes $\partial/\partial x$ to $\partial/\partial x$. So the vector field V exists.

b) On $\kappa_i(M_i)$, the flow is clearly given by $(t, \kappa_i^{-1}(x,y)) \mapsto \kappa_i^{-1}(t+x,y)$, wherever this makes sense. We then see that H_t takes $\kappa_0^{-1}(x,y)$ to $\kappa_0^{-1}(a,(-1)^n y)$ when $x+t = a+2\pi n$ with $n \in \mathbb{Z}$ and $a \in (-\pi,\pi)$ and to $\kappa_1^{-1}(b,(-1)^n y)$ when $x+t = b+2\pi n$ with $n \in \mathbb{Z}$ and $b \in (0,2\pi)$. The image of a point of $\kappa_1^{-1}M_1$ is similarly defined.

It follows that $H_{2\pi}$ sends $\kappa_0^{-1}(x,y)$ to $\kappa_0^{-1}(x,-y)$ and $\kappa_1^{-1}(x,y)$ to $\kappa_1^{-1}(x,-y)$. So $H_{2\pi}$ is not the idendity, but $H_{4\pi} = (H_{2\pi})^2$ is.

c) Suppose M is oriented by an orientation o. The diffeomorphism H_t takes o to an orientation $H_{t*}(o)$. Since an orientation can take only two values, and since $H_{t*}(o)$ is continuous in t, $H_{t*}(o)$ is constant in t. In particular, $H_{2\pi}$ is preserves the orientation. But at $\kappa_1(0,0)$ that is clearly not the case, in view of the above formula: we get a contradiction.

d) A diffeomorphism $S^1 \times (0,1) \to N$ is given as follows:

$$(e^{i\alpha}, y) \mapsto \begin{cases} \kappa_0(2\alpha, y) & \text{when } \alpha \in (-\pi/2, \pi/2); \\ \kappa_1(2\alpha, -y) & \text{when } \alpha \in (0, \pi); \\ \kappa_0(2\alpha - \pi, -y) & \text{when } \alpha \in (\pi/2, 3\pi/2); \\ \kappa_1(2\alpha - 2\pi, y) & \text{when } \alpha \in (\pi, 2\pi). \end{cases}$$

Since S^1 is orientable, so is N.

Opgave 4. Let $H : \mathbb{R} \times M \to M$ be a flow on a manifold M of dimension m and let the vector field V be its infinitesimal generator.

a) Let $f : M \to \mathbb{R}$. Prove that $\frac{\partial}{\partial t}|_{t=0} H_t^* f = V(f)$.

b) Let W be a vector field on M. Prove that $\frac{\partial}{\partial t}\Big|_{t=0} H_t^* W = [V, W]$.

c) Suppose that M is oriented. Prove that for every m-form μ on M with compact support, the integral $\int_M H_t^* \mu$ is independent of t.

Solution:

For the first two items see the notes. As to the last one, we must show that $\int_M H_t^* \mu$ is constant as a function of t. We do this by proving that its derivative is constant zero.

$$\frac{\partial}{\partial t}\Big|_{t=0} H_t^* \mu = \mathcal{L}_V \mu = \iota_V d\mu + d\iota_V \mu = d\iota_V \mu.$$

So

$$\frac{\partial}{\partial t}\Big|_{t=t_0} H_t^* \mu = \frac{\partial}{\partial t}\Big|_{t=0} H_t^*(H_{t_0}^* \mu) = d\iota_V(H_{t_0}^* \mu).$$

Hence

$$\frac{\partial}{\partial t}\Big|_{t=t_0} \int_M H_t^* \mu = \int_M \frac{\partial}{\partial t}\Big|_{t=t_0} \int_M H_t^* \mu = \int_M d\iota_V(H_{t_0}^* \mu) = 0$$

by Stokes' theorem.