Mid-term exam Mathematical Statistics

11 November 2004, 14.00-17.00 uur

Write your name and student number on each page you turn in. You may use all your lecture notes, the course literature and a simple calculator.

1. Let $(X_1, Y_1)^T, \ldots, (X_9, Y_9)^T$ be a sample from the normal distribution $N(\mu, \Sigma)$ with $\mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$. Let $Z_i = 3X_i + Y_i$, $V_i = 2X_i - 3Y_i$, $i = 1, \ldots, 7$, $\bar{X}_9 = \frac{1}{9}\sum_{i=1}^{9} X_i$, $\bar{Y}_9 = \frac{1}{9}\sum_{i=1}^{9} Y_i$, $\bar{Z}_7 = \frac{1}{7}\sum_{i=1}^{7} Z_i$, $\bar{V}_7 = \frac{1}{7}\sum_{i=1}^{7} V_i$, $S_y^2 = \frac{1}{8}\sum_{i=1}^{9}(Y_i - \bar{Y}_9)^2$, $S_z^2 = \frac{1}{6}\sum_{i=1}^{7}(Z_i - \bar{Z}_7)^2$, $S_v^2 = \frac{1}{6}\sum_{i=1}^{7}(V_i - \bar{V}_7)^2$.

   (a) Are X_1, Z_1 independent? Are Y_1, Z_1 independent? Find the joint distribution of $(Z_1, V_1)^T$. What is the joint distribution of $\big(\bar{V}_7, \frac{6S_v^2}{\mathrm{Var}(V_1)}\big)^T$? Does vector $(X_1, Z_1, V_1)^T$ have a density?

   (b) Compute $\frac{1}{5}E\bar{Z}_7$, $\frac{7}{3}\mathrm{Var}(\bar{Z}_7)$, $\frac{1}{2}ES_z^2$, $\mathrm{Var}\big(\frac{S_z^2}{\sqrt{3}}\big)$, $\frac{5}{43}ES_v^2$, $\mathrm{Var}(\frac{\sqrt{18}}{43}S_v^2)$.

   (c) Compute $\frac{21}{101}\mathrm{Var}(7\bar{Z}_7 - 3\bar{X}_9)$, $1 + \frac{1}{3}\mathrm{Var}(2S_y^2 - S_z^2)$, $1.8\mathrm{Cov}(X_1, V_1)$, $10 + \mathrm{Cov}(X_1, V_5)$ and $55P\big(\bar{Z}_7 > S_z\frac{0.906}{\sqrt{7}} + 5\big)$ (you may use here the relation $P(T \leq 0.906) = 0.8$ if $T \sim t_6$).

2. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent samples such that $EX_1 = EY_1 = \mu$ and $\mathrm{Var}(X_1) = \sigma^2$, $\mathrm{Var}(Y_1) = \alpha\sigma^2$, with known constant $\alpha > 0$. Define $\bar{X}_n = \sum_{i=1}^{n} X_i/n$, $\bar{Y}_m = \sum_{i=1}^{m} Y_i/m$.

   (a) Denote $T_1 = (n\bar{X}_n + m\bar{Y}_m)/(n + m)$ and $T_2 = (\alpha n\bar{X}_n + m\bar{Y}_m)/(m + \alpha n)$. Are these estimators unbiased for μ? Compute the MSE for both estimators. Which one is more preferable?

   (b) Suppose $m = m_n$ in such a way that $m_n/n \to 2$ as $n \to \infty$. Describe the asymptotic behaviour of T_1 and T_2 as $n \to \infty$ (if you have troubles, just let $m = 2n$). Determine the limit distribution of $\sqrt{n}\big(\sin(T_2) - \sin(\mu)\big)$ as $n \to \infty$.

   (c) Assume that X_1 and Y_1 are both normally distributed. Find the MLE for (μ, σ^2). Assume that σ^2 is known, then determine the Cramér-Rao lower bound for the estimation of μ and show that this bound is sharp. Assume that μ is known, then determine the Cramér-Rao lower bound for the estimation of σ^2 and show that this bound is sharp (you may use here the relation $\mathrm{Var}(Z^2) = 2\tau^4$ for $Z \sim N(0, \tau^2)$).

   (d) **(Extra)** Suppose $m = n$. Show that T_2 is the best estimator (in terms of MSE) among all unbiased estimators for μ which are linear combinations of $\bar{X}_n$ and $\bar{Y}_m$ (i.e. estimators of the form $\alpha\bar{X}_n + \beta\bar{Y}_m$, $\alpha, \beta \in \mathbb{R}$).

3. Let $X_1, \ldots, X_n$ be a sample form a shifted exponential distribution with the density $f_{\theta_1,\theta_2}(x) = \theta_1^{-1}e^{-(x-\theta_2)/\theta_1}I\{x \geq \theta_2\}$, where $\theta_1 > 0$ and $\theta_2 \in \mathbb{R}$. You may use here the fact that $X_1 \stackrel{d}{=} Y + \theta_2$ with $Y \sim \mathrm{Exp}(1/\theta_1)$ i.e. $Y \sim e^{-x/\theta_1}I\{x \geq 0\}/\theta_1$, $EY = \theta_1$, $\mathrm{Var}(Y) = \theta_1^2$.

   (a) Find the moment estimator $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2)$ for (θ_1, θ_2). Is it consistent? Assume that $\theta_1 + \theta_2 \neq 0$ and derive the limit distribution of $\sqrt{n}\big((\tilde{\theta}_1 + \tilde{\theta}_2)^{-1} - (\theta_1 + \theta_2)^{-1}\big)$ as $n \to \infty$.

   (b) Show that for any fixed $\theta_1 > 0$ the likelihood function is maximized at $\hat{\theta}_2 = X_{(1)} = \min\{X_1, \ldots, X_n\}$. Deduce that the joint MLE for (θ_1, θ_2) is given by $(\hat{\theta}_1, \hat{\theta}_2)$ with $\hat{\theta}_1 = \bar{X}_n - X_{(1)}$, $\bar{X}_n = \sum_{i=1}^{n} X_i/n$. Is the MLE unbiased? Is the MLE asymptotically unbiased?

   (c) Derive the limit distributions of $n(\hat{\theta}_2 - \theta_2)$ and $\sqrt{n}(\hat{\theta}_1 - \theta_1)$. Determine the limit distribution of $\sin(n(\hat{\theta}_2 - \theta_2))/\cos(n^{1/3}(\hat{\theta}_1 - \theta_1))$.

   (d) Assume that θ_2 is a known constant (you can take for example $\theta_2 = 1$). Compute the Cramér-Rao lower bound for the estimation of θ_1 and show that this bound is sharp.