Take-home second exam Mathematical Statistics, 5–14 February 2005

*Please write your name, your studentnumber and your e-mail address on your exam notes. * means a relatively difficult question, you may want to skip this question if you have troubles answering it. Your exam solutions should be returned to my postbox in the Mathematical Institute by Monday 14th February, 2005. Please note that I am expecting and trust you to work individually on these problems, no collaboration is allowed.*

1. Suppose we measure the diameter θ_1 of a circle: $X_i = \theta_1 + \xi_i$, $i = 1, \dots, n$, with independent measurement errors $\xi_i \sim N(0, \theta_2^2)$, $n \geq 2$, $\theta_1, \theta_2 > 0$ are unknown. Find the UMVU estimator of the area of the circle.

   *Describe the construction of the smallest $(1-\alpha)$-confidence interval for θ_2^2 based on the statistics $S_x^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$.

2. Let $X_1, \dots, X_n \sim N(\theta, \gamma^2\theta^2)$, where $\gamma > 0$ is known and $\theta > 0$ is unknown (consider also the case $\theta < 0$). Find a sufficient statistics for θ. Is it complete? Construct an exact (or at least asymptotic) $(1-\alpha)$-confidence interval for θ based on the statistics $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

3. Let a random sample $X_1, \dots, X_n$ be taken from a distribution that has the density $f_\theta(x) = \theta^{-1}e^{-x/\theta}I\{x \geq 0\}$, where $\theta > 0$ is unknown.

   Construct an asymptotic $(1-\alpha)$-confidence interval for θ.

   Construct the most powerful test for $H_0 : \theta = 1$ against $H_1 : \theta = 2$ of level $\alpha = 0.05$. Show that the test rejects H_0 if $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is sufficiently large, i.e. the critical region is of the form $K = \{\bar{X}_n \geq c_\alpha\}$. How can one find the critical value c_α for this test? Use the central limit theorem to find an approximate critical value for the most powerful test.

   *Find the UMVU estimator for $P(X \leq 2)$ if $X \sim f_\theta(x)$.

4. Suppose that a parameter θ takes on values $\theta_1 = 1$, $\theta_2 = 10$, $\theta_3 = 20$. The distributiomn of X is discrete and depends on θ as follows: $P_{\theta_1}(X = x_1) = P_{\theta_1}(X = x_2) = 0.1$, $P_{\theta_1}(X = x_3) = 0.2$, $P_{\theta_1}(X = x_4) = 0.6$; $P_{\theta_2}(X = x_1) = P_{\theta_2}(X = x_2) = P_{\theta_2}(X = x_3) = 0.2$, $P_{\theta_2}(X = x_4) = 0.4$; $P_{\theta_3}(X = x_1) = 0.4$, $P_{\theta_3}(X = x_2) = P_{\theta_3}(X = x_3) = P_{\theta_3}(X = x_4) = 0.2$. Assume a prior distribution on θ: $\pi(\theta = \theta_1) = 0.5$, $\pi(\theta = \theta_2) = 0.25$, $\pi(\theta = \theta_3) = 0.25$.

   Suppose that x_2 is observed. What are the posterior distribution and the Bayes estimator of θ for this case? Suppose that a second independent observation, x_1, is made. What does the posterior distribution become?

   *If one observation is made, determine the Bayes estimator for θ and its Bayes risk.

5. Let $X_1, \dots, X_n \sim \text{Poisson}(\mu)$, $\mu > 0$, and let μ have a $\text{Gamma}(r, \lambda)$ prior distribution. Determine the posterior distribution of μ and the Bayes estimator of μ. Is the Bayes estimator consistent? How does the Bayes estimator relate to the maximum likelihood estimator? Determine a $(1-\alpha)$-credible interval for μ.

6. Suppose X_1, X_2 two independent observations from the distribution $P(X_1 = \theta) = 1/2$, $P(X_1 = \theta + 1) = 1/2$, where $\theta \in \mathbb{R}$ is an uknown parameter. Construct a 75%-confidence interval with the smallest length. Does the notion of confidence interval really make sense in this situation? What can you suggest? Discuss this.

7. Let $X_1, \ldots, X_n$ be a sample from $N(\theta, \theta)$, with $\theta > 0$ unknown. We want to test H_0: $\theta = 1$ against H_1: $\theta > 1$, with significance level $\alpha = 0.05$.

   Which tests can you use for this problem? Describe all of them.

   Consider statistics $T = n(\bar{X}_n - 1)^2 + (n-1)S_x^2$. What distribution does statistics T have under the null hypothesis? Construct a test based on T.

   Suppose $n = 16$, the observed $\bar{x}_n = 1.45$, $s_x^2 = 1.55$. Apply all the tests you proposed to these data: which test does reject H_0, which does not? Compute also the corresponding p-values.

8. Let X be a random variable whose probability mass function under H_0 and H_1 is given as follows: $p_0(1) = p_0(2) = p_0(3) = p_0(4) = p_0(5) = p_0(6) = 0.01$, $p_0(7) = 0.94$; en $p_1(1) = 0.06$, $p_1(2) = 0.05$, $p_1(3) = 0.04$, $p_1(4) = 0.03$, $p_1(5) = 0.02$, $p_1(6) = 0.01$, $p_1(7) = 0.79$. Find the most powerful test for H_0 against H_1 of level $\alpha = 0.04$. Compute the probability of second type error for this test.

9. A die was cast $n = 120$ independent times and the following data resulted: 1 spot - b times, 2 spots - 20 times, 3 spots - 20 times, 4 spots - 20 times, 5 spots - 20 times, 6 spots - $40 - b$ times. If we use a chi-square test, for what values of b would the hypothetsis that the die is unbiased be rejected at the 0.025 significance level?