Utrecht University
Mathematics
Stochastic processes
Fall 2011

Test, November 8, 2011

JUSTIFY YOUR ANSWERS
Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

NOTE:

- The test consists of five questions plus one bonus problem.
- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. Let X_i, $1 = 1, \ldots, n$ be independent random variables. Consider

$$S = \sum_{i=1}^{n} X_i$$

Prove or disprove each of the following assertions:

(a) (1 pt.) If each X_i is a binomial random variable, then S also has a binomial law.

(b) (1 pt.) If each X_i is a exponential random variable, then S also has an exponential law.

Exercise 2. (2 pts.) Let X, Y be independent random variables with non-zero means. Prove that $Z = XY$ satisfies the identity

$$Z \;=\; \frac{E(Z \mid X)\, E(Z \mid Y)}{E(Z)}\;.$$

Exercise 3. Consider the following experiment: A number N of identical coins is chosen according to a Poisson law of rate λ and flipped. Each coin has probability p of turning head. Let X be the number of heads obtained.

(a) (1 pt.) Find $P(X = k)$ for each $k \in \mathbb{N}$.

(b) (1 pt.) Identify the law of X.

Exercise 4. A dance floor is lighted with blue and red lights which are randomly lighted. The color of each flash depends on the color of the two precedent flashes in the following way:

-i- Two consecutive red flashes are followed by another red flash with probability 0.9.

-ii- A red flash preceded by a blue flash is followed by a red flash with probability 0.6.

-iii- A blue flash is followed by a red flash with probability 0.5 if preceded by a red flash, and with probability 0.4 if preceded by another blue flash.

(a) (1 pt.) Write a transition matrix representing the process.

(b) (1 pt.) Find, in the long run, the proportion of time the dance floor is under red light.

(Turn over, please)

Exercise 5. Consider a Markov chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$\mathbb{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/4 & 0 & 1/2 & 1/2 \end{pmatrix}$$

(a) (1 pt.) Determine the classes of communicating states and specify whether they are recurrent or transient.

(b) (1 pt.) Determine all stationary distributions (also known as invariant measures).

Bonus problem

Bonus Consider a homogeneous (or shift-invariant) Markov chain $(X_n)_{n\in\mathbb{N}}$ and, for every state y, its first, second, ..., ℓ-th *hitting times*

$$\begin{aligned} T_y^{(1)} &= \min\{n > 0 : X_n = y\} \\ T_y^{(2)} &= \min\{n > T_y^{(1)} : X_n = y\} \\ &\vdots \\ T_y^{(\ell)} &= \min\{n > T_y^{(\ell-1)} : X_n = y\} . \end{aligned}$$

Let us denote

$$f_y^{(\ell)}(n) = P\big(T_y^{(\ell)} = n \mid X_0 = y\big) .$$

(a) (1 pt.) Prove that

$$f_y^{(2)}(n) = \sum_{i=1}^{n-1} f_y^{(1)}(i)\, f_y^{(1)}(n-i) ,$$

that is, $f_y^{(2)} = f_y^{(1)} * f_y^{(1)}$, where "$*$" stands for convolution.

(b) (1 pt.) State, and prove by induction, the corresponding formula for $f_y^{(\ell)}$ for any natural $\ell \geq 1$.