Utrecht University
Mathematics
Stochastic processes
Fall 2011

Test, January 17, 2012

JUSTIFY YOUR ANSWERS
Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

NOTE:

- The test consists of four problems plus two bonus problems
- The score is computed by adding all the credits up to a maximum of 10

Problem 1. Two bikers are riding at night. The mean lifetimes of the batteries of the front lights are respectively 4 and 6 hours (but riders can not tell which is which). One of the batteries has just been exhausted. Assuming that lifetimes are independent exponentially distributed random variables, compute

(a) (0.2 pts.) The probability that the exhausted battery be the one with larger mean lifetime.

(b) (0.8 pts.) The expected additional lifetime of the other battery.

Problem 2. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let T_n denote the n-th inter-arrival time and S_n the time of the n-th event. Let $t > 0$.

(a) Find:

-i- (0.4 pts.) $P\big(N(t) = 10, N(t/2) = 5, N(t/4) = 3\big)$.
-ii- (0.8 pts.) $E\big[S_5 \bigm| S_4 = 3\big]$.
-iii- (0.8 pts.) $E\big[T_2 \bigm| T_1 < T_2 < T_3\big]$.
-iv- (0.8 pts.) $E\big[N(t)\, N(t/2)\big]$.

(b) Prove:

-i- (0.4 pts.) $E\big[N(t) \bigm| N(t/2)\big] = N(t/2) + \lambda t/2$.
-ii- (0.8 pts.) $E\big[N(t/2) \bigm| N(t)\big] = N(t)/2$.

Problem 3. At an airport travellers arrive following a Poisson process with rate 1000/hour. As they go through customs, 10% of them are randomly chosen for a superficial inspection and a further 1% are chosen for a detailed inspection (that is, 11% of travellers are inspected).

(a) (0.4 pts.) Given that in the first ten minutes fifteen passengers have been submitted to the superficial inspection, what is the probability that in the same period exactly 4 passengers have gone through the detailed inspection.

(b) There are three officers performing superficial inspections and one performing detailed inspections. Inspection times are independent and exponentially distributed with a mean of 2 minutes for a superficial inspection and 5 minutes for a detailed one. A passenger, due for a detailed inspection, finds the corresponding inspector available while the three "superficial" inspectors are each of them busy with other travellers. Find the probability the passenger subjected to the detailed inspection

-i- (0.4 pts.) be the *first* to leave?
-ii- (1 pt.) be the *last* to leave?

Problem 4. Consider a birth-and-death process with three states (0,1 and 2). Such a process is characterized by the birth rates λ_0 and λ_1 and the death rates μ_1 and μ_2. All other rates are zero.

(a) (0.8 pts.) If the process starts at state zero, determine the expected time for it to reach state 2.

(b) (1 pt.) Write the 9 backward Kolmogorov equations, and observe that they form three sets of three coupled linear differential equations.

(c) (0.4 pts.) If $\lambda_0 = \mu_2 = \lambda$, prove that $P_{00}(t) - P_{20}(t) = \mathrm{e}^{-\lambda t}$.

(d) (1 pt.) If $\lambda_0 = \lambda_1 = \lambda$ and $\mu_1 = \mu_2 = \mu$ determine for which ratios λ/μ the system spends, in the long run, at least $1/3$ of the time in state 0.

Bonus problems

Only one of them may count for the grade

You can try both, but only the one with the highest grade will be considered

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Bonus 1. (1 pt.) Recall the "loss of memory" property of an exponential random variable X:

$$P\big(X > s+t\big) \;=\; P\big(X > s\big)\, P\big(X > t\big)\;.$$

for $s, t \in [0, \infty)$. Show that the property remains valid when s and t are replaced by independent random variables. That is, prove that if X is an exponentially distributed random variable and S and T two independent (say, continuous) non-negative random variables,

$$P\big(X > S+T\big) \;=\; P\big(X > S\big)\, P\big(X > T\big)\;.$$

Bonus 2. (1 pt.) Consider a continuous-time Markov chain with state space $S = \{0, 1, \ldots, n\}$, jump rates ν_i, $0 \le i \le n$, and transition probabilities P_{ij}, $0 \le i, j \le n$. Prove that a measure $(P_i)_{0 \le i \le n}$ on S is invariant for the continuous-time process if, and only if, the measure

$$\pi_i \;=\; \frac{\nu_i\, P_i}{\sum_j \nu_j\, P_j} \qquad 0 \le i \le n$$

is invariant for a discrete-time Markov process on S defined by the matrix $(P_{ij})_{0 \le i,j \le n}$.