Utrecht University
Mathematics
Stochastic processes
Fall 2012

Test, March 12, 2013

JUSTIFY YOUR ANSWERS
Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

NOTE:

- The test consists of five problems for a total of 11.5 points
- The score is computed by adding all the points up to a maximum of 10

Problem 1. Car inspections require, on average, 45 minutes. For Monday morning a garage has scheduled an inspection at 9 AM and another at 10 AM. Both car owners come on time. Assuming that the time required by successive inspections are IID exponentially distributed random variables, determine:

(a) (1 pt.) The expected waiting time for the owner of the 10 AM car before her inspection starts.

(b) (1 pt.) The expected amount of time the owner of the 10 AM car will spend in the garage.

Answers:

(a) Let T_1 be the service time of the 9AM car, T_2 the service time of the 10AM car and W the waiting time of the latter. We have

$$W \;=\; \begin{cases} 0 & \text{if } T_1 \le 1 \\ T_1 - 1 & \text{if } T_1 > 1 \end{cases} .$$

Hence, conditioning,

$$\begin{aligned} E(W) &= E\big(W \bigm| T_1 \le 1\big)\, P(T_1 \le 1) + E\big(W \bigm| T_1 > 1\big)\, P(T_1 > 1) \\ &= 0 + E\big(T_1 - 1 \bigm| T_1 - 1 > 0\big)\, \mathrm{e}^{-60/45} \\ &= 45\, \mathrm{e}^{-60/45}\, \text{min};. \end{aligned}$$

The last equality is due to the memoryless property of the exponential (as shown in class).

(b) Using (a),

$$E(W + T_2) \;=\; E(W) + E(T_2) \;=\; \big[45\, \mathrm{e}^{-60/45} + 45\big]\, \text{min} .$$

Problem 2. The arrival of customers to a shop is well approximated by a Poisson process $N(t)$ of rate λ. The i-th customer spends a random amount X_i, where the random variables $X_1, X_2, \ldots$ are independent and identically distributed with mean μ and variance σ^2. Let

$$Y(t) \;=\; \sum_{i=1}^{N(t)} X_i$$

be the revenue of the shop at time t.

(a) (1 pt.) Determine the mean revenue $E\big[Y(t)\big]$ for each $t > 0$.

(b) (1 pt.) Show that the variance of the revenue is $\mathrm{Var}\big[Y(t)\big] \;=\; \lambda\, t\, (\mu^2 + \sigma^2)$.

Answers:

(a) *We start by the tower property of conditional expectations:*

$$E\big[Y(t)\big] \;=\; E\Big(E\big[Y(t) \mid N(t)\big]\Big)\;. \tag{1}$$

We have, by the independence of $N(t)$ and the variables X_i,

$$\begin{aligned} E\big[Y(t) \mid N(t)=n\big] \;&=\; E\Big[\sum_{i=1}^{n} X_i \;\Big|\; N(t)=n\Big] \\ &=\; \sum_{i=1}^{n} E\big[X_i \mid N(t)=n\big] \\ &=\; \sum_{i=1}^{n} E\big[X_i\big] \\ &=\; n\,\mu\;. \end{aligned}$$

*Hence $E\big[Y(t) \mid N(t)\big] = \mu\, N(t)$ and, by (**??**),*

$$E\big[Y(t)\big] \;=\; \mu\, E\big[N(t)\big] \;=\; \mu\,\lambda\, t\;.$$

(b) *We have to compute $E\big[Y(t)^2\big]$. We condition as in (**??**):*

$$E\big[Y(t)^2\big] \;=\; E\Big(E\big[Y(t)^2 \mid N(t)\big]\Big)\;. \tag{2}$$

By the independence of $N(t)$ and the variables X_i,

$$E\big[Y(t)^2 \mid N(t)=n\big] \;=\; E\Big[\sum_{i,j=1}^{n} X_i\, X_j \;\Big|\; N(t)=n\Big] \;=\; \sum_{i,j=1}^{n} E\big[X_i\, X_j\big]\;.$$

At this point we must distinguish the case $i \neq j$ —involving independent variables X_i and X_j—- from the case $i = j$ for which $X_i\, X_j = X_i^2$. As there are $n(n-1)$ pairs (i,j) with $i \neq j$, we obtain

$$\begin{aligned} E\big[Y(t) \mid N(t)=n\big] \;&=\; \sum_{\substack{i,j=1\\ i\neq j}}^{n} E\big[X_i\big]\, E\big[X_j\big] + \sum_{i=1}^{n} E\big[X_i^2\big] \\ &=\; n(n-1)\,\mu^2 + n\,(\sigma^2+\mu^2) \\ &=\; n^2\,\mu^2 + n\,\sigma^2\;. \end{aligned}$$

*Hence, $E\big[Y(t)^2 \mid N(t)\big] = \mu^2\, N(t)^2 + \sigma^2\, N(t)$ and, by (**??**),*

$$\begin{aligned} E\big[Y(t)^2\big] \;&=\; \mu^2\, E\big[N(t)^2\big] + \sigma^2\, E\big[N(t)\big] \\ &=\; \mu^2\left(\mathrm{Var}\big[N(t)\big] + E\big[N(t)\big]^2\right) + \sigma^2\, E\big[N(t)\big] \\ &=\; \mu^2\left[\lambda\, t + (\lambda\, t)^2\right] + \sigma^2\,\lambda\, t\;. \end{aligned}$$

Using (a) we conclude

$$\mathrm{Var}\big[Y(t)\big] \;=\; E\big[Y(t)^2\big] - E\big[Y(t)\big]^2 \;=\; \mu^2\,\lambda\, t + \sigma^2\,\lambda\, t\;.$$

Problem 3. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let T_n denote the n-th inter-arrival time and S_n the time of the n-th event. Let $t > 0$. Find:

(a) (1 pt.) $P\big(N(t) = 10, N(t/2) = 5 \mid N(t/4) = 3\big)$.

(b) (1 pt.) $E\big[S_6 \mid S_4 = 3\big]$.

(c) (1 pt.) $E\big[T_3 \mid T_1 < T_2 < T_3\big]$.

Answers:

(a) By independence of $N(t) - N(t/2)$, $N(t/2) - N(t/4)$ *and* $N(t/4)$,

$$
\begin{aligned}
&P\big(N(t) = 10, N(t/2) = 5 \mid N(t/4) = 3\big)\\
&\quad = P\big(N(t) - N(t/2) = 5\,, N(t/2) - N(t/4) = 2 \mid N(t/4) = 3\big)\\
&\quad = P\big(N(t) - N(t/2) = 5\,, N(t/2) - N(t/4) = 2\big)\\
&\quad = P\big(N(t) - N(t/2) = 5\big)\, P\big(N(t/2) - N(t/4) = 2\big)\\
&\quad = \frac{(\lambda\, t)^5}{5!} \mathrm{e}^{-\lambda t}\, \frac{(\lambda\, t)^2}{2!} \mathrm{e}^{-\lambda t}\,.
\end{aligned}
$$

(b) As T_5, T_6 *and* S_4 *are independent random variables,*

$$
\begin{aligned}
E\big[S_6 \mid S_4 = 3\big] &= E\big[S_4 + T_5 + T_6 \mid S_4 = 3\big] \;=\; E\big[3 + T_5 + T_6 \mid S_4 = 3\big]\\
&= 3 + E\big[T_5 \mid S_4 = 3\big] + E\big[T_6 \mid S_4 = 3\big]\\
&= 3 + E\big[T_5\big] + E\big[T_6\big]\\
&= 3 + \frac{2}{\lambda}\,.
\end{aligned}
$$

(c) We must use the "good" random variables $\min\{T_1, T_2, T_3\}$ *and* $\Delta_i = T_i - \min\{T_1, T_2, T_3\}$*:*

$$
\begin{aligned}
E\big[T_3 \mid T_1 < T_2 < T_3\big] &= E\big[T_1 + \Delta_3 \mid T_1 = \min\{T_1, T_2, T_3\}\,,\, \Delta_2 < \Delta_3\big]\\
&= E_{T_1 = \min\{T_1, T_2, T_3\}}\big[T_1 + \Delta_3 \mid \Delta_2 < \Delta_3\big]\,.
\end{aligned}
$$

Now we use that, under the condition $T_1 = \min\{T_1, T_2, T_3\}$, *the variables* T_1, Δ_2 *and* Δ_3 *are independent and, furthermore,* $T_1 \sim \mathrm{Exp}(3\lambda)$ *and* $\Delta_2, \Delta_3 \sim \mathrm{Exp}(\lambda)$. *Hence,*

$$
\begin{aligned}
E\big[T_3 \mid T_1 < T_2 < T_3\big] &= E_{T_1 = \min\{T_1, T_2, T_3\}}\big[T_1\big] + E_{T_1 = \min\{T_1, T_2, T_3\}}\big[\Delta_3 \mid \Delta_2 < \Delta_3\big]\\
. &= \frac{1}{3\lambda} + \widetilde{E}\big[\Delta_3 \mid \Delta_2 < \Delta_3\big]\,,
\end{aligned}
\tag{3}
$$

where $\widetilde{E}$ *is an abbreviation of* $E_{T_1 = \min\{T_1, T_2, T_3\}}$. *We must again transcribe the remaining expectation in terms of "good" variables, in this case* $\min\{\Delta_2, \Delta_3\}$ *and* $\Delta_{32} = \Delta_3 - \Delta_2$*:*

$$
\begin{aligned}
\widetilde{E}\big[\Delta_3 \mid \Delta_2 < \Delta_3\big] &= \widetilde{E}_{\Delta_2 = \min\{\Delta_2, \Delta_3\}}\big[\Delta_2 + \Delta_{32}\big]\\
&= E_{\Delta_2 = \min\{\Delta_2, \Delta_3\}}\big[\Delta_2\big] + E_{\Delta_2 = \min\{\Delta_2, \Delta_3\}}\big[\Delta_{32}\big]\\
&= \frac{1}{2\lambda} + \frac{1}{\lambda}\,.
\end{aligned}
$$

*Replacing in (**??**),*

$$
E\big[T_3 \mid T_1 < T_2 < T_3\big] \;=\; \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} \;=\; \frac{11}{6\lambda}\,.
$$

Problem 4. Consider a pure death process with three states That is, a process whose only non-zero rates are the death rates μ_1 and μ_2.

(a) (1 pt.) Write the six non-trivial forward Kolmogorov equations.

(b) (1 pt.) Find $P_{ii}(t)$ for $i = 0, 1, 2$.

Answers:

(a)

$$\begin{aligned}
P'_{00}(t) &= 0 \\
P'_{10}(t) &= \mu_1 \left[P_{00}(t) - P_{10}(t)\right] \\
P'_{11}(t) &= -\mu_1 P_{11}(t) \\
P'_{21}(t) &= \mu_2 \left[P_{11}(t) - P_{21}(t)\right] \\
P'_{22}(t) &= -\mu_2 P_{22}(t) \\
P'_{20}(t) &= \mu_2 \left[P_{10}(t) - P_{20}(t)\right]
\end{aligned}$$

(b) The corresponding initial value problems and solutions are the following:

$$\left.\begin{aligned} P'_{00}(t) &= 0 \\ P_{00}(0) &= 1 \end{aligned}\right\} \Longrightarrow P_{00}(t) = 1$$

$$\left.\begin{aligned} P'_{11}(t) &= -\mu_1 P_{11}(t) \\ P_{11}(0) &= 1 \end{aligned}\right\} \Longrightarrow P_{11}(t) = \mathrm{e}^{-\mu_1 t}$$

$$\left.\begin{aligned} P'_{22}(t) &= -\mu_2 P_{22}(t) \\ P_{22}(0) &= 1 \end{aligned}\right\} \Longrightarrow P_{22}(t) = \mathrm{e}^{-\mu_2 t} .$$

Problem 5. A public job search service has a single desk and, for security reasons admits a maximum of two persons at each time. Potential applicants arrive at a Poisson rate of 4 per hour, and the successive service times are independent exponential random variables with mean equal to 10 minutes.

(a) (0.5 pts.) Write the system as a birth-and-death process with S=number of applicants present.

(b) (1 pt.) Determine the invariant measure (P_0, P_1, P_2) of this process.

(c) (0.5 pts.) Determine the average number of applicants present in the office.

(d) (0.5 pts.) What proportion of time is the clerk free to read the newspaper?

Answers:

(a) $S = \{0, 1, 2\}$, $\lambda_0 = \lambda_1 = 4\,\mathrm{hr}^{-1}$ *and* $\mu_1 = \mu_2 = 6\,\mathrm{hr}^{-1}$.

(b) The equations to be satisfied are

$$\begin{aligned}
4\,P_0 &= 6\,P_1 \\
4\,P_1 &= 6\,P_2 \\
P_0 + P_1 + P_2 &= 1 ,
\end{aligned}$$

which imply

$$(P_0, P_1, P_2) = \Big(\frac{9}{19}, \frac{6}{19}, \frac{4}{19}\Big) .$$

(c) $P_1 + 2\,P_2 = 14/19$.

(d) 9/19, *approximately 47% of the time.*