Utrecht University
Stochastic processes
WISB362
Winter 2016

Exam April 18, 2016

JUSTIFY YOUR ANSWERS
Allowed: calculator, material handed out in class and *handwritten* notes (*your handwriting*). NO BOOK IS ALLOWED

NOTE:

- The test consists of five exercises for a total of 10 credits plus two bonus problems for a maximum of 1.5 pts.
- The score is computed by adding all the valid credits up to a maximum of 10.

Exercise 1. (0.6 pts.) Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with mean μ and variance σ^2 and let N be an integer-valued random variable of mean λ independent of the previous ones. Define

$$S = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \ .$$

Determine $E(S^2)$ as a function of μ, σ^2 and λ.

Exercise 2. Consider a Markov chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$\mathbb{P} \ = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix}$$

(a) (0.4 pts.) Show that $\mathbb{P}^n_{4\,4} = (1/3)^n$.

(b) (0.6 pts.) Show that the state "4" is transient.

(c) (0.6 pts.) Let $T = \inf\{n > 0 : X_n \neq 4\}$ be the time it takes the process to exit "4" (for ever). Compute $E(T \mid X_0 = 4)$. [*Hint:* you may want to use that for a discrete random variable Z, $E[Z] = \sum_{k \geq 0} P(Z > k)$.]

(d) (0.6 pts.) Let $T_3 = \inf\{n > 0 : X_n = 3\}$ be the absorption time at state "3". Compute $P(T = T_3 \mid X_0 = 4)$, that is the probability that the process exist "4" only to be absorbed by "3".

(e) (0.6 pts.) Compute *all* the invariant measures of the process.

Exercise 3. Let X_1, X_2 and X_3 be independent exponential random variables with respective rates λ_1, λ_2 and λ_3. Compute:

(a) (0.6 pts.) $P\big(X_1 > X_2 + t \mid X_1 > t\big)$.

(b) (0.6 pts.) $P\big(X_1 > X_2 + t \mid X_2 > t\big)$.

(c) (0.6 pts.) $E\big(X_3 \mid X_1 < X_2 < X_3\big)$.

Exercise 4. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Find

(a) (0.6 pts.) $P\big(N(5) = 5\,, N(17) = 17\,, N(20) = 20\big)$.

(b) (0.6 pts.) $E\big[N(20) \mid N(17) = 17\,, N(5) = 5\big]$.

(c) (0.6 pts.) Determine λ such that

$$E\big[N(20) \mid N(17) = 17\big] \;=\; E\big[N(17) \mid N(20) = 40\big]\;.$$

Exercise 5. An atom subjected to electromagnetic radiation oscillates between its ground state G and two excited states E_1 and E_2. Measurements show that the time the atom remains in each excited state is exponentially distributed with mean 1/4 (picoseconds), after which the atom relaxes to the ground state. Once relaxed, the atom remains in the ground state an exponential time with mean 1. Due to its lower energy, the atom goes 3 times more often to the state E_1 than to E_2.

(a) (0.6 pts.) Model this evolution as a continuous-time Markov chain among the positions E_1, G and $E2$. That is, determine the abandoning rates ν_G, ν_{E_1} and ν_{E_2} and the transitions $\mathbb{P}_{ij}$ with $i, j = E_1, G, E_2$.

(b) (0.4 pts.) Can this process be interpreted as a birth-and-death process?

(c) (0.6 pts.) Determine, in the long run, the fraction of time spent by the atom in each of the three positions.

(d) (0.8 pts.) Write the 9 backward Kolmogorov equations, and observe that they form three sets of three coupled linear differential equations.

(e) (0.6 pts.) Determine $P_{E_1\,E_1}(t) - P_{E_2\,E_1}(t)$.

Bonus problems

Only one of them may count for the grade

You can try both, but only the one with the highest grade will be considered

---

Bonus 1. [Not all states can be transient] Consider a homogeneous (or shift-invariant) Markov chain $(X_n)_{n\in\mathbb{N}}$ $(X_n)_{n\in\mathbb{N}}$ with *finite* state space S. Let us recall that the *hitting time* of a state y is

$$T_y \;=\; \min\big\{n \geq 1 : X_n = y\big\}\;.$$

(a) If $\ell \leq n \in \mathbb{N}$, $x, y \in S$, prove the following

-i- (0.5 pt.)

$$P\big(X_n = y, T_y = \ell \mid X_0 = x\big) \;=\; P_{yy}^{n-\ell}\, P\big(T_y = \ell \mid X_0 = x\big)\;.$$

-ii- (0.5 pt.)

$$P_{xy}^n \;=\; \sum_{\ell=1}^{n} P_{yy}^{n-\ell}\, P\big(T_y = \ell \mid X_0 = x\big)\;.$$

(b) Conclude the following:

-i- (0.3 pt.) If every state is transient, then for every $x, y \in S$.

$$\sum_{n\geq 0} P_{xy}^{n} \;<\; \infty\;.$$

-ii- (0.2 pt.) The previous result leads to a contradiction with the stochasticity property of the matrix $\mathbb{P}$. Hence not all states can be transient.

Bonus 2. [Invariant probabilities are indeed invariant] Consider a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with countable state-space $S = \{x_1, x_2, \ldots\}$, waiting rates ν_i and embedded transition matrix P_{ij}, $i, j \geq 1$. Let $(P_i)_{i\geq 1}$ be an invariant probability distribution, that is, a family of positive numbers P_i satisfying $\sum_i P_i = 1$ and

$$\sum_{k:k\neq i} P_k\, \nu_k\, P_{ki} \;=\; \nu_i\, P_i$$

for all $i \geq 1$. Prove that if the process is initially distributed with the invariant law (P_i), this law is kept for the rest of the evolution. That is, prove that

$$P\big(X(0) = x_i\big) = P_i \quad\Longrightarrow\quad P\big(X(t) = x_i\big) = P_i$$

for all $t \geq 0$. *Suggestion:* Follow the following steps.

(i) (0.5 pt.) Show that if $P\big(X(0) = x_i\big) = P_i$, then

$$P\big(X(t) = x_j\big) \;=\; \sum_i P_i\, P_{ij}(t)\;.$$

(ii) (0.7 pt.) Use Kolmogorov *backward* equations to show that, as a consequence,

$$\frac{d}{dt} P\big(X(t) = x_j\big) \;=\; 0$$

for all $t \geq 0$.

(iii) (0.3 pt.) Conclude.