Utrecht University
Introductory mathematics for finance
WISB373
Winter 2013

Final exam
April 15, 2013

JUSTIFY YOUR ANSWERS
Allowed: Calculator, material handed out in class, *handwritten* notes (*your handwriting*)
BOOKS ARE NOT ALLOWED

NOTE:

- The test consists of five questions plus one bonus problem.
- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. [Reduction of risk] An investor needs a stock one year from now. The stock is worth 100 E today and in one year it is expected to be worth 160 E with probability p and 40 E with probability $1-p$. The investor decides to borrow money and buy a call option with a strike price of 100 E —at its fair price. The bank charges 10% yearly interest.

(a) (0.7 pts.) Determine, for each market scenario, the total amount payed by the investor *at the end of the year* to purchase the stock and cancel the debt.

(b) (0.3 pts.) Show that the spread between the maximal and minimal amounts is smaller than the spread between the actual values of the stock at the end of the year.

(c) (0.3 pts.) Show that the mean value paid is larger than the mean stock value at time 1, for some p.

Answers:

(a)

$$\widetilde{p} \;=\; \frac{100\cdot 1.1 - 40}{160-40} \;=\; \frac{7}{12}\,.$$

Hence, at the end of the year, the debt of the investor is

$$\begin{aligned} V_0\,R &= \widetilde{p}\,V_1(H) + \widetilde{q}\,V_1(T) \\ &= \frac{7}{12}\,60 + \frac{5}{12}\,0 \\ &= 35 \end{aligned}$$

If $\omega_1 = H$ the investor exercises the option and pays

$$V_0\,R + K \;=\; 35 + 100 \;=\; 135\,.$$

If $\omega_1 = T$ the investor does not exercise the option and pays

$$V_0\,R + S_1(T) \;=\; 35 + 40 \;=\; 75\,.$$

(b) The spread of the previous two amounts is $135 - 75 = 60$*, which is smaller than* $S_1(H) - S_1(T) = 160 - 40 = 120$.

(c) The mean value paid is

$$p\,135 + (1-p)\,75 \;=\; 75 + 60\,p$$

if the investor buys the option, while it is

$$p\,S_1(H) + (1-p)\,S_1(T) \;=\; 40 + 120\,p$$

otherwise. The former is larger if

$$75 + 60\,p - (40 + 120\,p) \;=\; 35 - 60\,p \;\geq\; 0$$

that is, if $p \leq 7/12$.

Exercise 2. [Discrete stochastic integral] Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration on a probability space. Let $(Y_n)_{n\geq 0}$, $(D_n)_{n\geq 0}$ and $(W_n)_{n\geq 0}$ adapted processes satisfying the linear system of equations

$$\begin{aligned} Y_0 &= W_0 \\ Y_{n+1} &= Y_n + D_n\,(W_{n+1} - W_n) \qquad \text{for } n = 0, 1, 2, \ldots \end{aligned}$$

(a) (0.7 pts.) Prove that

$$Y_n = W_0 + \sum_{\ell=1}^{n} D_{\ell-1}\big(W_\ell - W_{\ell-1}\big)$$

(b) Prove that If $(W_n)_{n\geq 0}$ is a martingale,

-i- (0.7 pts.) $(Y_n)_{n\geq 0}$ is a martingale.

-ii- (0.7 pts.) $(Y_n^2)_{n\geq 0}$ is a sub-martingale.

Answers:

(a) By induction in n. For $n = 0$ the expression is true by definition of Y_0. Assume true for n, then, by the inductive hypothesis,

$$\begin{aligned} Y_{n+1} &= Y_n + D_n\,(W_{n+1} - W_n) \\ &= W_0 + \sum_{\ell=1}^{n} D_{\ell-1}\big(W_\ell - W_{\ell-1}\big) + D_n\,(W_{n+1} - W_n) \\ &= W_0 + \sum_{\ell=1}^{n+1} D_{\ell-1}\big(W_\ell - W_{\ell-1}\big) \end{aligned}$$

(b)

$$\begin{aligned} E\big(Y_{n+1} \mid \mathcal{F}_n\big) &= E\big[Y_n + D_n\,(W_{n+1} - W_n) \mid \mathcal{F}_n\big] \\ &= Y_n + D_n\,\big[E(W_{n+1} \mid \mathcal{F}_n) - W_n\big]\,. \end{aligned} \tag{1}$$

Hence,

$$E(W_{n+1} \mid \mathcal{F}_n) = W_n \implies E(Y_{n+1} \mid \mathcal{F}_n) = Y_n\,.$$

(c) As $(Y_n)_{n\geq 0}$ is a martingale by (bi), the conditionned Jensen inequality implies that

$$E(Y_{n+1}^2 \mid \mathcal{F}_n) \geq E(Y_{n+1} \mid \mathcal{F}_n)^2 = Y_n^2\,.$$

Alternative:

$$\begin{aligned} E\big(Y_{n+1}^2 - Y_n^2 \mid \mathcal{F}_n\big) &= E\big(D_n^2(W_{n+1} - W_n)^2 + 2D_n(W_{n+1} - W_n) \mid \mathcal{F}_n\big) \\ &= E\big(D_n^2(W_{n+1} - W_n)^2 \mid \mathcal{F}_n\big) + 2D_n\,E\big(W_{n+1} - W_n\big) \mid \mathcal{F}_n\big) \\ &= E\big(D_n^2(W_{n+1} - W_n)^2 \mid \mathcal{F}_n\big) + 0 \\ &\geq 0\,. \end{aligned}$$

The second equality is due to the martingale property of (W_n) and the last inequality to the fact that $D_n^2(W_{n+1} - W_n)^2 \geq 0$.

Exercise 3. [American vs European I] Consider a stock with initial price S_0 =80E evolving as a binomial model with $u = 1.2$ and $d = 0.8$. Bank interest, however, fluctuates according to the evolution of the market: Initially is 10%, but it decreases to 5% if the last market fluctuation is "H" (otherwise it remains at 10%). An investor wishes to place a put option for two periods with strike price 80E.

(a) (0.7 pts.) Compute the risk-neutral probability.

(b) If the investor opts for an European put option,

-i- (0.9 pts.) Compute the fair price of the option.

-ii- (0.7 pts.) Determine the hedging strategy for the seller of the option.

(c) If the investor opts for an American option with $G_n = K - S_n$,

-i- (0.9 pts.) Compute the fair price of the option.

-ii- (0.7 pts.) Determine the optimal exercise times for the investor.

-iii- (0.7 pts.) Show that the process of discounted option values $\overline{V}_n$ is *not* a martingale.

Answers: *The asset price model is*

$$
\begin{array}{ccc}
 & & S_2(HH) = 115.2 \\
 & S_1(H) = 96 & \\
S_0 = 80 & & S_2(HT) = S_2(TH) = 76.8 \\
 & S_1(T) = 64 & \\
 & & S_2(TT) = 51.2
\end{array}
$$

and the interest growth process is:

$$
\begin{array}{cc}
 & R_1(H) = 1.05 \\
R_0 = 1.10 & \\
 & R_1(T) = 1.10
\end{array}
$$

(a)

$$
\begin{aligned}
\widetilde{p}_0 &= \frac{80 \cdot 1.10 - 64}{96 - 64} = 0.75 \\
\widetilde{p}_1(H) &= \frac{96 \cdot 1.05 - 76.8}{115.2 - 76.8} = 0.625 \\
\widetilde{p}_1(T) &= \frac{64 \cdot 1.10 - 51.2}{76.8 - 51.2} = 0.75 \, .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\widetilde{p}(HH) &= 0.75 \cdot 0.625 = 0.47 \\
\widetilde{p}(HT) &= 0.75 \cdot 0.375 = 0.28 \\
\widetilde{p}(TH) &= 0.25 \cdot 0.75 = 0.19 \\
\widetilde{p}(TT) &= 0.25 \cdot 0.25 = 0.06 \, .
\end{aligned}
$$

(b) The process of option values is

$$
\begin{array}{ccc}
 & & V_2(HH) = 0 \\
 & V_1(H) = \frac{0.375 \cdot 3.2}{1.05} = 1.14 & \\
V_0 = \frac{0.75 \cdot 1.14 + 0.25 \cdot 8.73}{1.10} = 2.76 & & V_2(HT) = V_2(TH) = 3.2 \\
 & V_1(T) = \frac{0.75 \cdot 3.2 + 0.25 \cdot 28.8}{1.10} = 8.73 & \\
 & & V_2(TT) = 28.8
\end{array}
$$

Hence

-i- $V_0 = 2.76$.

-ii-

$$
\begin{array}{cc}
 & \Delta_1(H) = \frac{0 - 3.2}{115.2 - 76.8} = -0.08 \\
\Delta_0 = \frac{1.14 - 8.73}{96 - 64} = -0.24 & \\
 & \Delta_1(T) = \frac{3.2 - 28.8}{76.8 - 51.2} = -1
\end{array}
$$

(c) *The intrinsic payoffs are:*

$$
\begin{array}{ccc}
 & & G_2(HH) = -35.2 \\
 & G_1(H) = -16 & \\
G_0 = 0 & & G_2(HT) = G_2(TH) = 3.2 \\
 & G_1(T) = 16 & \\
 & & G_2(TT) = 28.8
\end{array}
$$

Hence, the option values are

$$
\begin{array}{ccc}
 & & V_2(HH) = 0 \\
 & V_1(H) = \max\{-16, 1.14\} = 1.14 & \\
V_0 = \max\{0, \frac{0.75\cdot 1.14 + 0.25\cdot 16}{1.10}\} = 4.41 & & V_2(HT) = V_2(TH) = 3.2 \\
 & V_1(T) = \max\{16, 8.73\} = 16 & \\
 & & V_2(TT) = 28.8
\end{array}
$$

-*i*- $V_0 = 4.41$

-*ii*- *Using that* $\tau^* = \min\{n : V_n = G_n\}$, *we obtain*

$$
\begin{aligned}
\tau^*(T) &= 1 \\
\tau^*(HH) &= \infty \\
\tau^*(HT) &= 2
\end{aligned}
$$

-*iii*-

$$V_1(T) = 16 \ > \ E\Big(\frac{V_2}{R_1} \Bigm| \mathcal{F}_1\Big)(T) = 8.73$$

Exercise 4. [American vs European II] (0.7 pts.) Prove that, given the same market model and strike price, an American option with payoff G_n, $n = 0, \ldots, N$, can not be cheaper than a European option with final payoff G_N. Without loss of generality one can assume $G_n \geq 0$.

Answer: *Using the notation of the course (and the book)*

$$V_0^A \;=\; \max_{\tau \in \mathcal{S}_n} \widetilde{E}\Big[\mathbb{I}_{\{\tau \leq N\}} \frac{G_\tau}{R_0 \cdots R_{\tau-1}}\Big] \,.$$

As the stopping time $\tau = N$ *is among those in the right-hand side,*

$$V_0^A \;\leq\; \widetilde{E}\Big[\frac{G_N}{R_0 \cdots R_{N-1}}\Big] \;=\; V_0^E$$

Exercise 5. [Filtrations and (non-)stopping times] Two numbers are randomly generated by a computer. The only possible outcomes are the numbers 1, 2 or 3. The corresponding sample space is $\Omega_2 = \big\{(\omega_1, \omega_2) : \omega_i \in \{1, 2, 3\}\big\}$. Consider the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$, where $\mathcal{F}_0$ is formed only by the empty set and Ω_2, $\mathcal{F}_1$ formed by all events depending only on the first number, and $\mathcal{F}_2$ all events in Ω_2 (this is the ternary version of the two-period binary scenario discussed in class).

(a) (0.7 pts.) List all the events forming $\mathcal{F}_1$.

(b) (0.7 pts.) Let $\tau : \Omega_2 \longrightarrow \mathbb{N} \cup \{\infty\}$ defined as the "last outcome equal to 3". That is, $\tau(3, \omega_2) = 1$ if $\omega_2 \neq 3$, $\tau(\omega_1, 3) = 2$ for all ω_1, and $\tau = \infty$ if no 3 shows up. Prove that τ is *not* a stopping time with respect to the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$.

Answers:

(a) $\mathcal{F}_1 = \{\emptyset, B_1, B_2, B_3, B_{12}, B_{13}, B_{23}, \Omega_2\}$, *where*

$$\begin{aligned} B_i &= \big\{(i,1),(i,2),(i,3)\big\} \\ B_{ij} &= \big\{(i,1),(i,2),(i,3),(j,1),(j,2),(j,3)\big\}\ . \end{aligned}$$

(b) $\{\tau = 1\} = \big\{(3,1),(3,2)\big\} \notin \mathcal{F}_1$.

Bonus problem

Bonus. [Converse of exercise 2] (1.5 pts.) Let $(\mathcal{F}_n)_{n\ge 0}$ be the filtration defined by a binary market model and let $(Y_n)_{n\ge0}$ and $(W_n)_{n\ge0}$ two adapted processes with $Y_n(T) < Y_n < Y_n(H)$ and $W_n(T) < W_n < W_n(H)$ (as usual in the course, common arguments $\omega_1,\ldots,\omega_n$ are omitted from the notation). Prove that if both processes are martingales for a given measure —that is, for the same given p_n, q_n—, then one process is the stochastic integral of the other, that is, there exists an adapted process D_n such that

$$Y_n \;=\; Y_0 + \sum_{\ell=1}^{n} D_{\ell-1}\big(W_\ell - W_{\ell-1}\big) \tag{2}$$

Suggested steps:

(a) Show that the existence of p_n, $q_n = 1 - p_n$ such that

$$\begin{aligned} p_n\, Y_{n+1}(H) + q_n\, Y_{n+1}(T) &= Y_n \\ p_n\, W_{n+1}(H) + q_n\, W_{n+1}(T) &= W_n \end{aligned}$$

implies that there exist $\mathcal{F}_n$-measurable functions D_n such that

$$\frac{Y_{n+1}(H) - Y_n}{W_{n+1}(H) - W_n} \;=\; D_n \;=\; \frac{Y_n - Y_{n+1}(T)}{W_n - W_{n+1}(T)}\ . \tag{3}$$

(b) Deduce that

$$Y_{n+1} \;=\; Y_n + D_n\,(W_{n+1} - W_n) \qquad \text{for } n = 1,2,\ldots \tag{4}$$

(c) Conclude.

Answers: *I follow the proposed steps. As usual in the course, I am omitting common arguments* $\omega_1,\ldots,\omega_n$ *in the following discussion.*

(a) The identity $p_n\, Y_{n+1}(H) + (1-p_n)\, Y_{n+1}(T) = Y_n$ *implies*

$$p_n = \frac{Y_n - Y_{n+1}(T)}{Y_{n+1}(H) - Y_{n+1}(T)} \qquad \textit{and hence} \qquad q_n = \frac{Y_{n+1}(H) - Y_n}{Y_{n+1}(H) - Y_{n+1}(T)}\ .$$

Likewise, the identity $p_n\, W_{n+1}(H) + (1-p_n)\, W_{n+1}(T) = W_n$ *implies*

$$p_n = \frac{W_n - W_{n+1}(T)}{W_{n+1}(H) - W_{n+1}(T)} \qquad \textit{and hence} \qquad q_n = \frac{W_{n+1}(H) - W_n}{W_{n+1}(H) - W_{n+1}(T)}\ .$$

Equating the two expressions of p_n *we obtain*

$$\frac{Y_n - Y_{n+1}(T)}{W_n - W_{n+1}(T)} \;=\; \frac{Y_{n+1}(H) - Y_{n+1}(T)}{W_{n+1}(H) - W_{n+1}(T)}\ ,$$

while equating the two expressions of q_n yields

$$\frac{Y_{n+1}(H) - Y_n}{W_{n+1}(H) - W_n} \;=\; \frac{Y_{n+1}(H) - Y_{n+1}(T)}{W_{n+1}(H) - W_{n+1}(T)} \;.$$

These last two identities implies the proposed result (3) with

$$D_n \;=\; \frac{Y_{n+1}(H) - Y_{n+1}(T)}{W_{n+1}(H) - W_{n+1}(T)} \;.$$

(b) From (3)

$$\begin{aligned} Y_{n+1}(H) &= Y_n + D_n\,(W_{n+1}(H) - W_n) \quad \text{and} \\ Y_{n+1}(T) &= Y_n + D_n\,(W_{n+1}(T) - W_n)\;. \end{aligned}$$

This proves (4).

(c) Expression (2) follows by induction from (4), using the same argument as for Exercise 2(a).