Deeltentamen 1 Inleiding Financiele Wiskunde, 2011-12

* Punten per opgave:

opgave:	1	2	3
punten:	50	30	20

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.2$, $d = 0.9$, and $r = 0.1$. Suppose the real probability measure P satisfies $P(H) = p = \frac{1}{2} = P(T)$.

   (a) Consider an option with payoff $V_2 = (\max(S_1, S_2) - 100)^+$. Determine the price V_n at time $n = 0, 1$.

   (b) Suppose $\omega_1\omega_2 = TH$, find the values of the portfolio process $\Delta_0, \Delta_1(T)$ so that the corresponding wealth process satisfies $X_0 = V_0$ (your answer in part (a)) and $X_2(TH) = V_2(TH)$.

   (c) Suppose a trader is selling the above option for a price $T > V_0$. Explain how the trader can perform arbitrage, i.e. with begin wealth equals to zero he can build a portfolio that has at time 2 a non-negative value with probability 1.

   (d) Determine explicitly the Radon-Nikodym process Z_0, Z_1, Z_2, where

   $$Z_2(\omega_1\omega_2) = Z(\omega_1\omega_2) = \frac{\widetilde{P}(\omega_1\omega_2)}{P(\omega_1\omega_2)}$$

   with $\widetilde{P}$ the risk neutral probability measure, and $Z_i = E_i(Z)$, $i = 0, 1,$.

   (e) Consider the utility function $U(x) = \sqrt{x}$ $(x > 0)$. Show that the random variable $X = X_2$ (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = X_0$ is given by

   $$X = X_2 = \frac{(1.1)^2 X_0}{Z^2 E(Z^{-1})}.$$

   (f) Assume in part (e) that $X_0 = 100$. Determine the value of the optimal portfolio process $\{\Delta_0, \Delta_1\}$ and the value of the corresponding wealth process $\{X_0, X_1, X_2\}$.

2. Consider the N-period Binomial model with risk neutral probability measure $\widetilde{P}$. Suppose $X_0, X_1, \cdots, X_N$ is an adapted process satisfying $X_i > -1$ for all $i = 0, 1, \cdots, N$. Define a process $Y_0, Y_1, \cdots, Y_N$ by

   $$Y_0 = 1, \quad \text{and } Y_n = \frac{1}{(1 + X_0) \cdots (1 + X_{n-1})}, \; n = 1, \cdots, N.$$

(a) Let $U_n = \widetilde{E}_n \left[\frac{Y_N}{Y_n}\right]$, $n = 0, 1, \cdots, N$. Show that the process $Y_0U_0, Y_1U_1, \cdots, Y_NU_N$ is a martingale with respect to $\widetilde{P}$.

(b) Let $\Delta_0, \cdots, \Delta_{N-1}$ be an adapted process, and W_0 a fixed positive real number. Define for $n = 0, 1, \cdots, N-1$,

$$W_{n+1} = \Delta_n U_{n+1} + (1 + X_n)(W_n - \Delta_n U_n).$$

Show that the process

$$Y_0W_0, Y_1W_1, \cdots, Y_NW_N$$

is a martingale with respect to $\widetilde{P}$.

(c) Let U_n be as given in part (a). Set $I_0 = 0$ and define $I_n = \sum_{j=0}^{n-1} Y_{j+1}(U_{j+1} - U_j)$, $n = 1, \cdots, N$. Show that $I_0, I_1, \cdots, I_N$ is a martingale with respect to $\widetilde{P}$.

3. Consider the N-period binomial model, with expiration process N. Let $\widetilde{P}$ be the risk neutral probability and P the real probability. We denote by $p = P(H)$ and $\widetilde{p} = \widetilde{P}(H)$. Let $S_0, S_1, \cdots, S_N$ be the corresponding price process.

(a) Define $Y_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k$. Show that the process

$$(S_0, Y_0), (S_1, Y_1), \ldots, (S_N, Y_N)$$

is Markov with respect to P and $\widetilde{P}$.

(b) Let $V_N = (Y_N - S_N)^+$. Show that for each $n = 0, 1, \cdots, N$, there exists a function f_n such that

$$E_n(ZV_N) = Z_n(1 + r)^{N-n} f_n(S_n, Y_n),$$

where Z is the Radon-Nikodym derivative of $\widetilde{P}$ with respect to P, and $Z_n = E_n(Z)$, $n = 0, 1, \cdots, N$.