Universiteit Utrecht | 35 pt | Boedapestlaan 6
Mathematisch Instituut | | 3584 CD Utrecht

Deeltentamen 2 Inleiding Financiele Wiskunde, 2011-12

exercise:	1	2	3	4
points:	25	25	25	25

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.1$, $d = 0.8$, and $r = 0.05$. Consider an American Put option with expiration $N = 2$ and strike price $K = 90$.

   (a) Determine the price V_n at time $n = 0, 1$ of the American put option.

   (b) Determine the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$.

   (c) Suppose $\omega_1\omega_2 = TT$. Find the values of the replicating portfolio process $\Delta_0, \Delta_1(T)$. Show that if the buyer exercises at time 1, then $X_1(T) = V_1(T)$, and if the buyer exercises at time 2, then $X_2(TT) = V_2(TT)$.

2. Consider the binomial model with up factor $u = 2$, down factor $d = 1/2$ and interest rate $r = 1/4$. Consider a perpetual American put option with $S_0 = 2^j$, and $K = S_0 2^{-m}$. Suppose that the buyer of the option exercises the first time the price is less than or equal to $K/2$.

   (a) Show that the price at time zero of this option is given by

   $$V_0 = \begin{cases} K - S_0, & \text{if } S_0 \leq K/2, \\ \dfrac{K^2}{4S_0}, & \text{if } S_0 \geq K. \end{cases}$$

   (b) Consider the process $v(S_0), v(S_1), \cdots$ defined by

   $$v(S_n = \begin{cases} K - S_n, & \text{if } S_n \leq K/2, \\ \dfrac{K^2}{4S_n}, & \text{if } S_n \geq K. \end{cases}$$

   Show that $v(S_n) \geq (K - S_n)^+$ for all $n \geq 0$, and that the discounted process $\{\left(\dfrac{4}{5}\right)^n v(S_n) : n = 0, 1, \cdots\}$ is a supermartingale.

3. Consider a random walk $M_0, M_1, \cdots$ with probability p for an up step and $q = 1 - p$ for a down step, $0 < p < 1$. For $a \in \mathbb{R}$, define $S_n^a = 10^{-n+aM_n}$, $n = 0, 1, 2, \cdots$.

   (a) For which values of a is the the process $S_0^a, S_1^a, \cdots$ a martingale?

   (b) Suppose now that $p = 1/2$, so $M_0, M_1, \cdots$, is the symmetric random walk. Let $\tau_m = \inf\{n \geq 0 : M_n = m\}$. Determine the value of $E(S_{\tau_m}^a)$.

4. Consider a 3-period (non constant interest rate) binomial model with interest rate process R_0, R_1, R_2 defined by

$$R_0 = 0,\ R_1(\omega_1) = .05 + .01H_1(\omega_1),\ R_2(\omega_1, \omega_2) = .05 + .01H_2(\omega_1, \omega_2)$$

where $H_i(\omega_1, \cdots, \omega_i)$ equals the number of heads appearing in the first i coin tosses $\omega_1, \cdots, \omega_i$. Suppose that the risk neutral measure is given by $\widetilde{P}(HHH) = \widetilde{P}(HHT) = 1/8$, $\widetilde{P}(HTH) = \widetilde{P}(THH) = \widetilde{P}(THT) = 1/12$, $\widetilde{P}(HTT) = 1/6$, $\widetilde{P}(TTH) = 1/9$, $\widetilde{P}(TTT) = 2/9$.

(a) Calculate $B_{1,2}$ and $B_{1,3}$, the time one price of a zero coupon maturing at time two and three respectively.

(b) Consider a 3-period interest rate swap. Find the 3-period swap rate SR_3, i.e. the value of K that makes the time zero no arbitrage price of the swap equal to zero.

(c) Consider a 3-period floor that makes payments $F_n = (.055 - R_{n-1})^+$ at time $n = 1, 2, 3$. Find Floor_3, the price of this floor.