Universiteit Utrecht 35 pt Boedapestlaan 6
Mathematisch Instituut 3584 CD Utrecht

Oefen Deeltentamen 2 Inleiding Financiele Wiskunde, 2011-12

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.2$, $d = 0.7$, and $r = 0.1$. Consider now an Asian American put option with expiration $N = 2$, and intrinsic value $G_n = 95 - \frac{S_0 + \cdots + S_n}{n+1}$, $n = 0, 1, 2$.

   (a) Determine the price V_n at time $n = 0, 1$ of this American option.

   (b) Find the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$.

   (c) Suppose it is possible to buy this option at a price $C > V_0$, where V_0 is your answer from part (a). Construct an explicit arbitrage strategy.

2. Let $M_0, M_1, \cdots,$ be the symmetric random walk, i.e. $M_0 = 0$, and $M_n = \sum_{i=1}^n X_i$, where

$$X_i = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T, \end{cases}$$

   for $i \geq 1$. Let $m \geq 2$ be an integer, and let $k \in \{1, \cdots, m-1\}$. Define $Y_0 = k$, and

$$Y_{n+1} = (Y_n + X_{n+1})\mathbb{I}_{\{Y_n \notin \{0,m\}\}} + Y_n \mathbb{I}_{\{Y_n \in \{0,m\}\}},$$

   for $n \geq 0$.

   (a) Show that $Y_0, Y_1, \cdots$ is a martingale.

   (b) Let $T = \inf\{n \geq 1 : Y_n \in \{0, m\}\}$. Using the the Optional Sampling Theorem show that $E(Y_T) = E(Y_0) = k$.

   (c) Prove that $P(Y_T = 0) = \frac{m-k}{m}$.

3. Consider the binomial model with up factor $u = 2$, down factor $d = 1/2$ and interest rate $r = 1/4$. Consider a perpetual American put option with $S_0 = 2^j$, and $K = S_0 2^{-m}$. Suppose that the buyer of the option exercises the first time the price is less than or equal to $K/2$.

   (a) Show that the price at time zero of this option is given by

$$V_0 = \begin{cases} K - S_0, & \text{if } S_0 \leq K/2, \\ \frac{K^2}{4S_0}, & \text{if } S_0 \geq K. \end{cases}$$

   (b) Consider the process $v(S_0), v(S_1), \cdots$ defined by

$$v(S_n = \begin{cases} K - S_n, & \text{if } S_n \leq K/2, \\ \frac{K^2}{4S_n}, & \text{if } S_n \geq K. \end{cases}$$

Show that $v(S_n) \geq (K - S_n)^+$ for all $n \geq 0$, and that the discounted process $\{\left(\frac{4}{5}\right)^n v(S_n) : n = 0, 1, \cdots\}$ is a supermartingale.

4. Consider a 3-period (non constant interest rate) binomial model with interest rate process R_0, R_1, R_2 defined by

$$R_0 = 0,\ R_1(\omega_1) = 0.02 f(\omega_1),\ R_2(\omega_1, \omega_2) = 0.02 f(\omega_1) f(\omega_2)$$

where $f(H) = 3$, and $f(T) = 2$. Suppose that the risk neutral measure is given by $\widetilde{P}(HHH) = \widetilde{P}(HTT) = 1/10$, $\widetilde{P}(HHT) = \widetilde{P}(HTH) = 1/5$, $\widetilde{P}(THH) = \widetilde{P}(THT) = 1/15$, $\widetilde{P}(TTH) = \widetilde{P}(TTT) = 2/15$.

   (a) Calculate the time one price $B_{1,3}$ of a zero coupon bond with maturity $m = 3$.

   (b) Consider a 3-period interest rate swap. Find the 3-period swap rate SR_3, i.e. the value of K that makes the time zero no arbitrage price of the swap equal to zero.

   (c) Consider a 3-period Cap that makes payments $C_n = (R_{n-1} - 0.1)^+$ at time $n = 1, 2, 3$. Find Cap_3, the price of this Cap.