Utrecht University
Introductory mathematics for finance
WISB373
Winter 2013

Exam, May 22, 2013

JUSTIFY YOUR ANSWERS
Allowed: Calculator, material handed out in class, *handwritten* notes (*your handwriting*)
BOOKS ARE NOT ALLOWED

NOTE:

- The test consists of five exercises for a total of 12 credits.
- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. (Payments and interest) You subscribe a loan to be payed in monthly instalments at an effective yearly interest rate r per year during N years. You have completed $n \leq 12N$ payments. Find:

(a) (0.8 pts.) The part of the principal reimbursed so far.

(b) (0.8 pts.) The total amount payed so far in interest.

Answers: *We use formulas developed in Problem 2 of the first problem set.*

(a) The part of the principal repaid in the k-th payment is

$$P\,\frac{r}{12}\,\frac{\big(1+\frac{r}{12}\big)^{k-1}}{\big(1+\frac{r}{12}\big)^{12N}-1}\,.$$

Hence, the principal reimbursed in the first n payments is

$$P\,\frac{r}{12}\,\frac{\sum_{k=1}^{n}\big(1+\frac{r}{12}\big)^{k-1}}{\big(1+\frac{r}{12}\big)^{12N}-1} \;=\; P\frac{\big(1+\frac{r}{12}\big)^{n}-1}{\big(1+\frac{r}{12}\big)^{12N}-1}\,.$$

(b) The interest paid in the k-th payment is

$$P\,\frac{r}{12}\,\frac{\big(1+\frac{r}{12}\big)^{12N}-\big(1+\frac{r}{12}\big)^{k-1}}{\big(1+\frac{r}{12}\big)^{12N}-1}\,.$$

Hence, the interest paid at the end of n payments is

$$P\,\frac{r}{12}\,\frac{n\,\big(1+\frac{r}{12}\big)^{12N}-\sum_{k=1}^{n}\big(1+\frac{r}{12}\big)^{k-1}}{\big(1+\frac{r}{12}\big)^{12N}-1} \;=\; P\,\frac{r}{12}\,\frac{n\,\big(1+\frac{r}{12}\big)^{12N}}{\big(1+\frac{r}{12}\big)^{12N}-1} - P\frac{\big(1+\frac{r}{12}\big)^{n}-1}{\big(1+\frac{r}{12}\big)^{12N}-1}\,.$$

Exercise 2. (Die without glasses) You throw a dice but you can only detect the parity of the outcome. That is, you have access only to the σ-algebra $\mathcal{F}_E = \big\{\emptyset, \{\text{even outcome}\}, \{\text{odd outcome}\}, \Omega\big\}$. Determine which of the following functions $X : \Omega = \{1,2,3,4,5,6\} \to \mathbb{R}$ are measurables with respect to $\mathcal{F}_E$:

(a) (0.8 pts.) $X(i) = i$.

(b) (0.8 pts.) $X(i) = (-1)^i$.

Answers: *More explicitly,*

$$\mathcal{F}_E = \big\{\emptyset, \{2,4,6\}, \{1,3,5\}, \Omega\big\}$$

(a) The function is not $\mathcal{F}_E$-measurable, for instance because $X^{-1}(\{1\}) = \{1\} \notin \mathcal{F}_E$.

(b) For X to be $\mathcal{F}_E$-measurable it must be of the form

$$X(\omega) \;=\; \begin{cases} c_1 & \text{if } \omega = 2,4,6 \\ c_2 & \text{if } \omega = 1,3,5 \end{cases}$$

for constants c_1 and c_2. But X is precisely of this form, with $c_1 = 1$ and $c_2 = -1$. Hence X is $\mathcal{F}_E$-measurable.

Exercise 3. (Martingales and submartingales) A biased coin, with a probability p of showing head, is repeatedly tossed. Let $(\mathcal{F}_n)$ be the filtration of the binary model, in which $\mathcal{F}_n$ are the events determined by the first n tosses. A stochastic process (X_j) is defined such that

$$X_j \;=\; \begin{cases} 1 & \text{if } j\text{-th toss results in head} \\ -1 & \text{if } j\text{-th toss results in tail} \end{cases} \qquad \text{for } j = 1,2,\ldots$$

Consider the process

$$\begin{aligned} M_0 &= 1 \\ M_n &= \exp\Big[\sum_{j=1}^{n} X_j\Big] \quad j \geq 1 \,. \end{aligned}$$

(a) (0.9 pts.) For which value of p is (M_n) a martingale adapted to the filtration $(\mathcal{F}_n)$?

(b) (0.9 pts.) For this value of p prove that the process

$$L_n \;=\; \mathrm{e}^{M_n}$$

is a submartingale adapted to the filtration $(\mathcal{F}_n)$.

Answers:

(a) The process (M_n) is adapted because (X_n) is. In order to be a martingale, the process should satisfy

$$E(M_{n+1} \mid \mathcal{F}_n) \;=\; M_n \qquad \textit{for all } n \geq 0 \,. \tag{1}$$

We compute:

$$E(M_{n+1} \mid \mathcal{F}_n) \;=\; E\Big(\prod_{j=0}^{n+1} \mathrm{e}^{X_j} \,\Big|\, \mathcal{F}_n\Big) \;=\; \Big(\prod_{j=0}^{n} \mathrm{e}^{X_j}\Big) E\big(\mathrm{e}^{X_{n+1}} \mid \mathcal{F}_n\big) \;=\; \Big(\prod_{j=0}^{n} \mathrm{e}^{X_j}\Big) E\big(\mathrm{e}^{X_{n+1}}\big) \,. \tag{2}$$

The third equality is due to the $\mathcal{F}_n$-measurability of $\prod_{j=0}^{n} \mathrm{e}^{X_j}$ ("taking out what is known") and the last equality is due to the independence of X_{n+1} with respect to $\mathcal{F}_n$. From (1) and (2) we see that (M_n) is a martingale if, and only if,

$$1 \;=\; E\big(\mathrm{e}^{X_j}\big) \;=\; p\,\mathrm{e} + (1-p)\,\mathrm{e}^{-1} \,.$$

Hence

$$p \;=\; \frac{1-\mathrm{e}^{-1}}{\mathrm{e}-\mathrm{e}^{-1}} \;=\; \frac{1}{\mathrm{e}+1} \,.$$

(b) By the conditioned Jensen inequality ($x \to \mathrm{e}^x$ is a convex function) and the martingale character of (M_n),

$$E(L_{n+1} \mid \mathcal{F}_n) \;=\; E\big(\mathrm{e}^{M_{n+1}} \mid \mathcal{F}_n\big) \;\geq\; \exp\Big[E(M_{n+1} \mid \mathcal{F}_n)\Big] \;=\; \mathrm{e}^{M_n} \;=\; L_n \,.$$

Exercise 4. (Asian option) Consider a stock with initial price S_0 whose price, at the end of each period, has a probability p of growing 20% and a probability $1-p$ of decreasing 20%. Bank interest is 10% for each period. An investor needs the stock at the end of three periods and wishes to pay at most S_0 at that time.

The investor considers an Asian call option with strike value S_0, that is an option that can only be exercised at the end of the third period, with payoff

$$V_3 = \left| \frac{1}{4} \sum_{j=0}^{4} S_j - S_0 \right|_+ .$$

For the evolution over 3 periods compute:

(a) (1 pt.) The risk-neutral probability

(b) (1 pt.) The initial price V_0 of the option.

(c) (1 pt.) The hedging strategy Δ_n $(n = 0, 1, 2)$ of the seller.

(d) (0.8 pts.) The average *net market* payoff as a function of p. That is, the market average of the payoff minus the initial payment translated to the end of the 3rd period. For which values of p it is on the average convenient for the investor to purchase the option.

Answers:

(a) The risk-neutral conditional probabilities are defined by

$$\widetilde{p} = \frac{1+r-d}{u-d} = \frac{1.1-0.8}{1.2-0.8} = \frac{3}{4} \qquad \widetilde{q} = 1 - \widetilde{p} = \frac{1}{4} .$$

We have:

$$\widetilde{P}(HHH) = \widetilde{p}^3 = \frac{27}{64}$$

.

$$\widetilde{P}(HHT) = \widetilde{P}(HTH) = \widetilde{P}(THH) = \widetilde{p}^2\widetilde{q} = \frac{9}{64}$$

.

$$\widetilde{P}(HTT) = \widetilde{P}(TTH) = \widetilde{P}(THT) = \widetilde{p}\widetilde{q}^2 = \frac{3}{64}$$

.

$$\widetilde{P}(TTT) = \widetilde{q}^3 = \frac{1}{64} .$$

(b) The only non-zero payoffs are

$$\begin{aligned} V_3(HHH) &= \frac{1}{4}\Big[1 + \frac{6}{5} + \Big(\frac{6}{5}\Big)^2 + \Big(\frac{6}{5}\Big)^3\Big] S_0 - S_0 = \frac{171}{500} S_0 = 0.342 S_0 \\ V_3(HHT) &= \frac{1}{4}\Big[1 + \frac{6}{5} + \Big(\frac{6}{5}\Big)^2 + \Big(\frac{6}{5}\Big)^2 \cdot \frac{4}{5}\Big] S_0 - S_0 = \frac{99}{500} S_0 = 0.198 S_0 \\ V_3(HTH) &= \frac{1}{4}\Big[1 + \frac{6}{5} + \frac{6}{5}\cdot\frac{4}{5} + \Big(\frac{6}{5}\Big)^2 \cdot \frac{4}{5}\Big] S_0 - S_0 = \frac{36}{500} S_0 = 0.072 S_0 . \end{aligned}$$

Hence, the initial value of the option is

$$V_0 = \frac{1}{1.1^3}\Big[\frac{27}{64}\cdot\frac{171}{500} + \frac{9}{64}\cdot\Big(\frac{99}{500} + \frac{36}{500}\Big)\Big] S_0 = \frac{5832}{42592} S_0 = 0.137\ldots S_0 . \tag{3}$$

(c) *Iterating backwards the relation*

$$V_{n+1} \;=\; \frac{1}{1.1}\Big[\frac{3}{4}V_n(H) + \frac{1}{4}V_n(T)\Big]$$

we get the following option values:

$$V_3(HHH) = \tfrac{171}{500}S_0$$

$$V_2(HH) = \tfrac{612}{2200}S_0$$

$$V_3(HHT) = \tfrac{99}{500}S_0$$

$$V_1(H) = \tfrac{1944}{9680}S_0$$

$$V_3(HTH) = \tfrac{36}{500}S_0$$

$$V_2(HT) = \tfrac{108}{2200}S_0$$

$$V_3(HTT) = 0$$

$$V_0 = \tfrac{5832}{42592}S_0$$

$$V_3(THH) = 0$$

$$V_2(TH) = 0$$

$$V_3(THT) = 0$$

$$V_1(T) = 0$$

$$V_3(TTH) = 0$$

$$V_2(TT) = 0$$

$$V_3(TTT) = 0$$

The asset values, on the other hand, are computed by the following formula

$$S_n(H^kT^{n-k}) \;=\; \Big(\frac{3}{4}\Big)^k\Big(\frac{1}{4}\Big)^{n-k}$$

for $k, n = 0, 1, 2, 3,\ k \le n$. *Using the formula*

$$\Delta_n \;=\; \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}$$

we get the following hedging policies:

$$\Delta_2(HH) = \tfrac{4608}{9000}$$

$$\Delta_1(H) = \tfrac{8069}{13200}$$

$$\Delta_2(HT) = \tfrac{2304}{3000}$$

$$\Delta_0 = \tfrac{1944}{4890}$$

$$\Delta_2(TH) = 0$$

$$\Delta_1(T) = 0$$

$$\Delta_2(TT) = 0$$

(d) *The net market payoff is*

$$Y(p) \;=\; p^3V_3(HHH) + p^2(1-p)\big[V_3(HHT) + V_3(HTH)\big] - 1.1^3V_0\,. \tag{4}$$

However, by (3),

$$1.1^3V_0 \;=\; \widetilde{p}^3V_3(HHH) + \widetilde{p}^2(1-\widetilde{p})\big[V_3(HHT) + V_3(HTH)\big]\,. \tag{5}$$

Formulas (4)–(5) show that:

- $Y(0) < 0$;
- $Y(\widetilde{p}) = 0$;
- $Y(1) > 0$, *and*
- $Y'(p) > 0$ *for* $0 \le p \le 1$.

We conclude that $Y(p)$ is a strictly increasing function that is positive only for $\widetilde{p} \leq p \leq 1$. In this interval the purchase of the option is advantageous. Otherwise, in the average, the investor will be better off not buying the option and paying the final market value of the stock.

Exercise 5. (Asian American option) In the same setup as in the previous exercise, the investor is offered, as an alternative, the American version of the preceding option. This is an option that can be exercised at the end of any period, and offers intrinsic payoff.

$$G_n = \frac{1}{n+1}\sum_{j=0}^{n} S_j - S_0 \qquad n = 1,2,3\,.$$

Let us call such an option an "Asian American option".

(a) (0.4 pts.) A theorem was discussed in class proving that the optimal exercise time for some American call options is at the last period or never, so they end up being no different than the European version. Explain why this theorem does not apply for the Asian American option.

(b) For this Asian American option:

- -i- (1 pt.) Compute the initial price V_0.
- -ii- (1 pt.) Compute the optimal exercise times for the investor.
- -iii- (0.8 pts.) Verify the validity of the formula

$$V_0 = \max_{\tau\in\mathcal{S}_0} \widetilde{\mathbb{E}}\Big[\mathbb{I}_{\{\tau\leq N\}}\frac{G_\tau}{(1+r)^\tau}\Big]\,.$$

Answers:

(a) The theorem requires intrinsic payoffs of the form $G_n = g(S_n)$. This is not the case here, because each intrinsic payoff depends also of previous asset values: $G_n = g_n(S_0,\ldots,S_n)$.

(b) We use the "American algorithm" to compute the value of the option at each period. Let us write it in the form

$$V_n = \max\{G_n, V_n^E\}$$

with

$$V_n^E = \widetilde{p}V_{n+1}(H) + \widetilde{q}V_{n+1}(T)\,.$$

The calculations are summarised in the following table:

$$V_3(HHH) = G_3(HHH)$$
$$V_3(HHH) = \frac{171}{500} S_0$$

$$V_2^E(HH) = \frac{612}{2200} S_0$$
$$G_2(HH) = \frac{16}{75} S_0$$
$$V_2(HH) = V_2^E(HH)$$

$$V_3(HHT) = G_3(HHT)$$
$$V_3(HHT) = \frac{99}{500} S_0$$

$$V_1^E(H) = \frac{1944}{9680} S_0$$
$$G_1(H) = \frac{1}{10} S_0$$
$$V_1(H) = V_1^E(H)$$

$$V_3(HTH) = G_3(HTH)$$
$$V_3(HTH) = \frac{36}{500} S_0$$

$$V_2^E(HT) = \frac{108}{2200} S_0$$
$$G_2(HT) < 0$$
$$V_2(HT) = V_2^E(HT)$$

$$V_3(HTT) = 0$$
$$G_3(HTT) < 0$$

$$V_0 = \frac{5832}{42592} S_0$$

$$V_3(THH) = 0$$
$$G_3(THH) < 0$$

$$V_2^E(TH) = 0$$
$$G_2(TH) < 0$$
$$V_2(TH) = V_2^E(TH)$$

$$V_3(THT) = 0$$
$$G_3(THT) < 0$$

$$V_1^E(T) = 0$$
$$G_1(T) < 0$$
$$V_1(T) = V_1^E(T)$$

$$V_3(TTH) = 0$$
$$G_3(TTH) < 0$$

$$V_2^E(TT) = 0$$
$$G_2(TT) < 0$$
$$V_2(TT) = V_2^E(TT)$$

$$V_3(TTT) = 0$$
$$G_3(TTT) < 0$$

-i- The previous table shows that the price of the Asian American option coincides with that of the Asian option.

-ii- The formula $\tau = \inf\{n : V_n = G_n\}$ *yields:*

$$\begin{aligned} \tau(HHH) = \tau(HHT) = \tau(HTH) &= 2 \\ \tau(THH) = \tau(TTH) = \tau(THT) = \tau(HTT) = \tau(TTT) &= \infty\,. \end{aligned}$$

-iii- The only finite exercise times are those for the evolutions HHH, HHT *and* HTH, *for which they take the value 2. Hence the proposed formula coincides with the calculation (3).*