Utrecht University
Introductory mathematics for finance
WISB373
Winter 2014

Exam, April 15, 2014

JUSTIFY YOUR ANSWERS
Allowed: Calculator, material handed out in class, *handwritten* notes (*your handwriting*)
NOT ALLOWED: Books, printed or photocopied material

NOTE:

- The test consists of five exercises plus one bonus problem.
- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. [Retirement plan] You subscribe to the following retirement plan. Each month, C euros are deducted from your salary and deposited in the retirement agency. Once your n-th deposit is completed, you have a choice of collecting a lump sum or receiving equal monthly payments for n consecutive months. Assume that the year interest is constant and equal to r

(a) (0.5 pts.) Show that, if you opt for a final lump sum, the retirement agency must return you, immediately after your n-th payment, an amount $C\,V(n, r/12)$, with

$$V(n,r) \;=\; \frac{(1+r)^n - 1}{r}\;.$$

(b) (0.5 pts.) Show that if, instead, you prefer to be payed monthly for n months, starting from the $(n+1)$-th month, the monthly payment you receive is

$$D \;=\; C\left(1 + \frac{r}{12}\right)^n\;.$$

Exercise 2. [Blurred die] You throw a dice but you can only detect the parity of the outcome. That is, you have access only to the σ-algebra $\mathcal{F}_E = \big\{\emptyset, \{2,4,6\}, \{1,3,5\}, \Omega\big\}$.

(a) Determine which of the following functions $F, G : \Omega = \{1,2,3,4,5,6\} \to \mathbb{R}$ are measurable with respect to $\mathcal{F}_E$:

-i- (0.5 pts.) $F(i) = (-1)^i$.

-ii- (0.5 pts.) $G(i) = (-2)^i$.

(b) Compute

-i- (0.5 pts.) $E(F \mid \mathcal{F}_E)$.

-ii- (0.5 pts.) $E(G \mid \mathcal{F}_E)$.

Exercise 3. [Martingales and submartingales] An unbiased coin (that is, with equal probability of showing heads and tails) is repeatedly tossed. Let $(\mathcal{F}_n)$ be the filtration of the binary model, in which $\mathcal{F}_n$ are the events determined by the first n tosses. A stochastic process (X_j) is defined such that

$$X_j \;=\; \begin{cases} \;\;1 & \text{if } j\text{-th toss results in head} \\ -1 & \text{if } j\text{-th toss results in tail} \end{cases} \qquad \text{for } j = 1, 2, \ldots$$

(Turn over, please)

(a) Prove that the process

$$\begin{aligned} Y_0 &= 0 \\ Y_n &= \sum_{j=1}^{n} X_j \quad j \geq 1 \,, \end{aligned}$$

-i- (0.6 pst.) is a martingale adapted to the filtration $(\mathcal{F}_n)$, and

-ii- (0.6 pts.) is Markovian

(b) (0.6 pts.) Prove that the process

$$\begin{aligned} M_0 &= 1 \\ M_n &= \exp\Bigl[\sum_{j=1}^{n} X_j\Bigr] \quad j \geq 1 \,. \end{aligned}$$

is a sub-martingale adapted to the filtration $(\mathcal{F}_n)$.

Exercise 4. [Asian European option] Consider a stock with initial price S_0 whose price, at the end of each period, has a probability p of growing 20% and a probability $1-p$ of decreasing 20%. Bank interest is 10% for each period. An investor needs the stock at the end of three periods and wishes to pay at most S_0 at that time. The investor considers an Asian that at the end of the third period has a payoff

$$V_3 = \Bigl| \max_{0 \leq j \leq 3} S_j - S_0 \Bigr|_+ \,.$$

For the evolution over 3 periods compute:

(a) (0.6 pts.) The risk-neutral probability

(b) (0.6 pts.) The initial price V_0 of the option.

(c) (0.5 pts.) The Radon-Nykodým derivative process Z_n, $n = 0, 1, 2, 3$.

Exercise 5. [Asian American option] In the same setup as in the previous exercise, the investor is offered, as an alternative, the American version of the preceding option. This is an option that can be exercised at the end of any period, and offers intrinsic payoff.

$$G_n = \max_{0 \leq j \leq n} S_j - S_0 \qquad n = 1, 2, 3 \,.$$

Let us call such an option an "Asian American option".

(a) (0.5 pts.) A theorem was discussed in class proving that the optimal exercise time for some American call options is at the last period or never, so they end up being no different than the European version. Explain why this theorem does not apply for the Asian American option.

(b) Compute:

(Next page, please)

-i- (0.6 pts.) The initial price V_0.

-ii- (0.6 pts.) The optimal exercise times for the investor.

-iii- (0.6 pts.) The hedging strategy of the financial institution.

-iv- (0.5 pts.) The consumption process C_n, $n = 0, 1, 2, 3$.

(c) (0.7 pts.) Verify the validity of the formula

$$V_0 \;=\; \max_{\tau \in \mathcal{S}_0} \widetilde{\mathbb{E}}\Big[\mathbb{I}_{\{\tau \le N\}} \frac{G_\tau}{(1+r)^\tau}\Big] \;.$$

Bonus problem

Bonus. [Dividend-paying stock] Consider the general binary (not necessarily binomial) dividend-paying stock model. The model is defined by stock prices S_n and growth factors R_n, $n = 0, \ldots, N$. At the end of each period, after the new stock value is attained, a dividend is paid and the stock price is reduced by the corresponding amount Formally, these operations are described by the following adapted non-negative random variables

(a) $Y_n(\omega_1, \ldots, \omega_n)$ representing the percentual change in stock value from time t_{n-1}^+ to t_n^-, that is, *before* paying dividend at t_n. Hence, the stock value at t_n^- is

$$S_n^- \;=\; Y_n S_{n-1} \;.$$

(b) $A_n(\omega_1, \ldots, \omega_n)$ representing the percent of the t_n^--value of the stock paid as a dividend at t_n^+. Thus,

$$S_n \;=\; (1 - A_n) Y_n S_{n-1} \;.$$

If the financial institution adopts hedging strategies Δ_n, the wealth equation for the values X_n of its portfolio becomes

$$X_{n+1} \;=\; \Delta_n Y_{n+1} S_n + R_n (X_n - \Delta_n S_n) \;.$$

Consider the risk-neutral measure defined by (omitting, as done in class, the overall dependence on $\omega_1, \ldots, \omega_n$)

$$\widetilde{p}_n \;=\; \frac{R_n S_n - S_{n+1}^-(T)}{S_{n+1}^-(H) - S_{n+1}^-(T)} \;=\; \frac{R_n - Y_{n+1}(T)}{Y_{n+1}(H) - Y_{n+1}(T)} \;.$$

Show the following:

(a) (0.5 pts) $\widetilde{E}(Y_{n+1} \mid \mathcal{F}_n) \;=\; R_n$.

(b) (0.5 pts) The discounted wealth process $\overline{X}_n$ is a $\widetilde{P}$-martingale, whichever the hedging strategy.

(c) (0.5 pts) The discounted stock price $\overline{S}_n$ is *not* a $\widetilde{P}$-martingale, but only a $\widetilde{P}$-super-martingale.

(d) (0.5 pts) In contrast, the process

$$\widehat{S}_n \;=\; \frac{\overline{S}_n}{(1 - A_1) \cdots (1 - A_n)}$$

is a martingale.