Universiteit Utrecht
Mathematisch Instituut

Boedapestlaan 6
3584 CD Utrecht

Mid-Term: Inleiding Financiele Wiskunde 2019-2020
Start at 9 am and stop writing at 11 am
Make good fotos of your exam and e-mail it to k.dajani1@uu.nl together with the signed Honor code

(1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise independent sets in $\mathcal{F}$ (i.e. $\mathbb{P}(A_n \cap A_m) = \mathbb{P}(A_n)\mathbb{P}(A_m)$ for $n \neq m$) satisfying $\mathbb{P}(A_n) = 1/2$ for all $n \geq 1$. Let $\mathbb{I}_{A_n}$ be the indicator function of the set A_n and $\sigma(\mathbb{I}_{A_n})$ the σ-algebra generated by the random variable $\mathbb{I}_{A_n}$, $n \geq 1$.

(a) Prove that $\sigma(\mathbb{I}_{A_n}) = \{\emptyset, \Omega, A_n, A_n^c\}$ and that the σ-algebras $\sigma(\mathbb{I}_{A_n})$ and $\sigma(\mathbb{I}_{A_m})$ are independent whenever $n \neq m$, i.e. $\mathbb{P}(C \cap D) = \mathbb{P}(C)\mathbb{P}(D)$ for any $C \in \sigma(\mathbb{I}_{A_n})$ and any $D \in \sigma(\mathbb{I}_{A_m})$. Conclude that $\mathbb{I}_{A_1}, \mathbb{I}_{A_2}, \cdots$ is a **pairwise independent** sequence. (1.5 pts)

(b) For $n \geq 1$, define $X_n = 2\mathbb{I}_{A_n} - 1$. Set $M_0 = 0$, $M_n = \sum_{k=1}^{n} 2^{k-1} X_k$ for $n \geq 1$ and let $Y_n = M_n^2 - \frac{(4^n - 1)}{3}$ for $n \geq 0$. Consider the filtration $\{\mathcal{F}(n) : n \geq 0\}$ where $\mathcal{F}(0) = \{\emptyset, \Omega\}$ and $\mathcal{F}(n) = \sigma(\mathbb{I}_{A_1}, \cdots, \mathbb{I}_{A_n})$ = the smallest σ-algebra containing all sets of the form $\{\mathbb{I}_{A_j} \in B\}$ for any Borel set B and any $1 \leq j \leq n$. Prove that the process $\{Y_n : n \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(n) : n \geq 0\}$. (1.5 pts)

(2) Let $\{W(t) : t \geq 0\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : t \geq 0\}$ be a filtration for the Brownian motion. Define a process $\{X(t) : t \geq 0\}$ by $X(t) = e^{tW(t) - t^3 + 1}$, $t \geq 0$.

(a) Prove that $\mathbb{P}(X(1) > 1) = 1/2$. (1 pt)

(b) Derive an expression for $\text{Var}[X(t)]$, the variance of $X(t)$. (1.5 pts)

(c) For $s < t$, determine an expression for $\mathbb{E}[X(t)|\mathcal{F}(s)]$. (1.5 pts)

(3) Let $\{W(t) : t \geq 0\}$ and $\{V(t) : t \geq 0\}$ be two **independent** Brownian motions defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By independence we mean that $W(t)$ and $V(s)$ are independent for all $s, t > 0$. Let $0 < \rho < 1$ be a positive real number and define a process $\{Z(t) : t \geq 0\}$ by $Z(t) = \rho\, W(t) + \sqrt{1 - \rho^2}\, V(t)$. Prove that the process $\{Z(t) : t \geq 0\}$ is a Brownian motion. (3 pts)

(**Hint**: if X and Y are independent normally distributed random variables with X being $\mathcal{N}(\mu_1, \sigma_1^2)$ and Y being $\mathcal{N}(\mu_2, \sigma_2^2)$, then $X + Y$ is normally $\mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ distributed).