Final: Inleiding Financiele Wiskunde 2019-2020

(1) Let $\{W(t) : t \geq 0\}$ be a Brownian motion with filtration $\{\mathcal{F}(t) : t \geq 0\}$. Consider the process $\{S(t) : t \geq 0\}$ defined by

$$S(t) = -\int_0^t 2S(u)\,du + \int_0^t e^{-4u}\,dW(u).$$

(a) Show that the process $\{e^{2t}S(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. (1 pt)

(b) Determine the distribution of $S(t)$. (1 pt)

(2) Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the price process $\{S(t) : t \geq 0\}$ given by

$$S(t) = 1 + \int_0^t \alpha S(u)\,dW_1(u) + \int_0^t \beta S(u)\,dW_2(u)$$

where α, β are positive constants.

(a) Show that $\{S^2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (1pt)

(b) Show that $\mathbb{E}[S^2(t)] = e^{(\alpha^2+\beta^2)t}$, $t \geq 0$. (You are allowed to interchange integrals and expectations but justify why). (1.5 pts)

(3) Let T fe finite horizon and let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$, where $\mathcal{F}(T) = \mathcal{F}$. Suppose that the price process $\{S(t) : 0 \leq t \leq T\}$ of a certain stock is given by

$$S(t) = \exp\Big\{\int_0^t (1+u)\,dW(u) + t - \frac{t^3}{6}\Big\}$$

(a) Show that $\{S(t) : 0 \leq t \leq T\}$ is an Itô-process. (1 pt)

(b) Let r be a constant interest rate. Find a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the discounted process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under $\widetilde{\mathbb{P}}$. (1.5 pts)

(4) Let T be a finite time (expiration date) and let $\{(W_1(t), W_2(t) : 0 \leq t \leq T)$ be a two-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$, where $\mathcal{F} = \mathcal{F}(T)$. Consider two price processes

$$dS_1(t) = S_1(t)\,dt + 0.3S_1(t)\,dW_1(t) + 0.3S_1(t)\,dW_2(t)$$
$$dS_2(t) = 2S_2(t)\,dt + 0.1S_2(t)\,dW_1(t).$$

We assume $S_1(0), S_2(0) > 0$.

(a) Assume that the interest rate is a constant, i.e. $R(t) = r$ for $t > 0$. Find the unique risk-neutral probability $\widetilde{\mathbb{P}}$, i.e. the probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ under which the discounted price processes $\{e^{-rt}S_i(t) : 0 \leq t \leq T\}$ are martingales, $i = 1, 2$. (1.5 pts)

(b) Consider a financial derivative with payoff at time T given by $V(T) = \dfrac{1}{T}\displaystyle\int_0^T S_2(t)\,dt$. Show that the *fair* price at time 0 of this derivative is given by $V(0) = \frac{S_2(0)}{rT}(1 - e^{-rT})$. (1.5 pts)