bag = [text width=3cm, text centered] end = [] Utrecht University
Mathematical Institute

Mid-Term Exam for Introduction to Financial Mathematics, WISB373

Friday May 21th 2021, 13:15 - 15:15 **(2 hours examination)**

1. Flip a biased coin three times with $\mathbb{P}(H) = \frac{1}{4}$ and $\mathbb{P}(T) = \frac{3}{4}$. So our probability space is $(\Omega, \mathcal{F}, \mathbb{P})$, with

$$\Omega = \{HHH; HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

$\mathcal{F}$ is the power set of Ω, and

$$\begin{aligned} \mathbb{P}(HHH) &= \frac{1}{64}, \quad \mathbb{P}(TTT) = \frac{27}{64}. \\ \mathbb{P}(HHT) &= P(HTH) = P(THH) = \frac{3}{64}, \\ \mathbb{P}(HTT) &= P(THT) = P(TTH) = \frac{9}{64}. \end{aligned}$$

Let $\mathcal{F}_1$ be the σ-algebra containing the information on the first coin flip, i.e., $\mathcal{F}_1 = \sigma(\{A_H, A_T\})$, with $A_H = \{HHH, HHT, HTH, HTT\}$ and $A_T = \{THH, THT, TTH, TTT\}$. Define X on Ω by

$$X = 16 \cdot \mathbb{1}_{\{HHH;HHT\}} + 8 \cdot \mathbb{1}_{\{HTH,HTT,THH,THT\}} + 4 \cdot \mathbb{1}_{\{TTH,TTT\}}.$$

a. Find an explicit expression for $\mathbb{E}[X|\mathcal{F}_1]$. (1 pt)

b. Define the price process S_0, S_1, S_2, S_3 on Ω by a tree, with $S_0 = 4$ and three coin tosses. Each time a head is tossed we have $S_i = 2S_{i-1}$, and each time a tail is obtained, we have $S_i = \frac{1}{2}S_{i-1}$. Draw the corresponding tree, and show that $\sigma(S_2) \neq \mathcal{F}_2$. ($\mathcal{F}_2$ is the sigma algebra that contains the information about the first two coin flips.) (2 pt)

2. Let $\{W(t) : t \geq 0\}$ be a Brownian motion, we define a process $\{X(t) : t \geq 0\}$ by

$$X(t) = \frac{1}{\sqrt{3}} W(3t).$$

a. Prove that $\{X(t) : t \geq 0\}$ is a Brownian motion. (1 pt)

b. Let $Y(t) = X^2(t) - 2\sqrt{c}t$ for some non-negative constant c and for all $t \geq 0$. For which value of c is the process $\{Y(t) : t \geq 0\}$ a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$, with $\mathcal{F}(t) = \sigma(X(s) : s \leq t)$? (1 pt)

3. Suppose $\{W(t) : t \geq 0\}$ is a Brownian Motion, $\{\mathcal{F}(t) : t \geq 0\}$ is a filtration for $\{W(t) : t \geq 0\}$ and $\sigma > 0$. The Geometric Brownian Motion, GBM, $\{S(t) : 0 \leq t \leq T\}$ is defined by

$$S(t) = S(0) \exp \left\{ \sigma W(t) - \frac{1}{2}\sigma^2 t \right\}$$

with $\{W(t) : t \geq 0\}$ a BM. This process can be used to model certain asset prices, where parameter σ is the volatility.

The log-return on the interval $[t_i, t_{i+1}]$ is defined as,

$$\log \left(\frac{S(t_{i+1})}{S(t_i)} \right).$$

a. Show that, on the $0 \leq T_1 \leq T_2$, with the partition

$$T_1 = t_0 < t_1 < \ldots < t_m = T_2,$$

the quadratic variation of the log-returns gives us an estimate for the realized volatility in the time interval $[T_1, T_2]$. (2 pt)

b. Show that $S(t)$ is a martingale under the filtration $\mathcal{F}(t)$. (2 pt)