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In 2004/2005, the course WISM424 was given by J. van Oosten.

Set Theory (WISM424)
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Advice: start on those problems you can do right away; then, start thinking about the others. Good luck!

Exercise 1

We recall the cardinal numbers $\beth_\alpha$ (for each ordinal number α), recursively defined by: $\beth_0 = \omega$; $\beth_{\alpha+1} = 2^{\beth_\alpha}$; $\beth_\gamma = \sup\{\beth_\beta \mid \beta < \gamma\}$ if γ is a limit ordinal.
A cardinal κ is called a *strong limit* if κ is uncountable and for all $\lambda, \mu < \kappa$, $\lambda^\mu < \kappa$.
Show that the following three conditions are equivalent for a cardinal κ:

a) κ is a strong limit;

b) For all $\lambda < \kappa$, $2^\lambda < \kappa$;

c) There is a limit ordinal $\alpha > 0$ such that $\kappa = \beth_\alpha$

Exercise 2

A *Luzin set* is an uncountable subset L of $\mathbb{R}$ such that for every closed and nowhere dense subset A of $\mathbb{R}$, $L \cap A$ is countable.

a) Show that the collection of all closed and nowhere dense subsets of $\mathbb{R}$ has cardinality 2^ω.

b) Assuming the Continuum Hypothesis, show that a Luzin set exists.
[Hint: use an enumeration $\{K_\alpha \mid \alpha < \omega_1\}$ of the closed nowhere dense subsets of $\mathbb{R}$. Use the Baire category theorem, which states that the union of countably many closed nowhere dense sets has empty interior]

Exercise 3

a) Prove that the intersection of two clubs (closed, unbounded subsets of ω_1) is again a club.

b) An ordinal $\alpha < \omega_1$ is called a *limit of limits* if there is a strictly increasing sequence $\gamma_0 < \gamma_1 < \cdots$ of limit ordinals such that $\alpha = \sup\{\gamma_n \mid n < \omega\}$.
Prove that the set of all limits of limits is a stationary subset of ω_1.

Exercise 4

In this exercise, M is a countable transitive model of ZFC and P is a poset in M. M^P is the set of P-names in M.

a) Suppose that $A \epsilon M$ is an antichain in P and for each $q \epsilon A$ a name $\sigma_q \epsilon M^P$ is given such that the sequence $(\sigma_q)_{q\epsilon A}$ is in M. Define the following P-name:

$$\pi = \bigcup_{q\epsilon A} \{(\tau, r) \mid r \leq q \wedge (r \Vdash \tau \epsilon \sigma_q) \wedge (\tau \epsilon \mathrm{dom}(\sigma_q))\}$$

Show that for every $q \epsilon A$, $q \Vdash \pi = \sigma_q$.

b) Let $\phi(x, y)$ be a ZF-formula, $\tau \epsilon M^P$ and $p \epsilon P$. Suppose $p \Vdash \exists y \phi(y, \tau)$. Show that there is a $\pi \epsilon M^P$ such that $p \Vdash \phi(\pi, \tau)$.
[Hint: show that there is a subset A of P which is maximal with respect to the properties that $A \epsilon M$, $\forall a \epsilon A\,(a \leq p)$, A is an antichain in P, and $\forall a \epsilon A \exists \sigma_a \epsilon M^P\,(a \Vdash \phi(\sigma_a, \tau))$. Apply part a).]