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Final Ergodic Theory

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1. Consider $([0,1),\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Lebesgue σ-algebra, and λ is Lebesgue measure. Let $\beta > 1$ be a real number satisfying $\beta^3 = \beta^2+\beta+1$, and consider the β-transformation $T_\beta : [0,1)\to[0,1)$ given by $T_\beta x = \beta x \pmod 1$. Define a measure ν on $\mathcal{B}$ by

$$\nu(A) = \int_A h(x)\,dx\,,$$

where

$$h(x) = \begin{cases} \frac{1}{\frac{1}{\beta}+\frac{2}{\beta^2}+\frac{3}{\beta^3}}\left(1+\frac{1}{\beta}+\frac{1}{\beta^2}\right) & \text{if } x\in[0,1/\beta) \\ \frac{1}{\frac{1}{\beta}+\frac{2}{\beta^2}+\frac{3}{\beta^3}}\left(1+\frac{1}{\beta}\right) & \text{if } x\in[1/\beta, 1/\beta+1/\beta^2) \\ \frac{1}{\frac{1}{\beta}+\frac{2}{\beta^2}+\frac{3}{\beta^3}}.1 & \text{if } x\in[1/\beta+1/\beta^2,1)\ , \end{cases}$$

   (a) Show that T_β is measure preserving with respect to ν.

   (b) Let

$$X = \left([0,\frac{1}{\beta})\times[0,1)\right)\times\left([\frac{1}{\beta},\frac{1}{\beta}+\frac{1}{\beta^2})\times[0,\frac{1}{\beta}+\frac{1}{\beta^2})\right)\times\left([\frac{1}{\beta}+\frac{1}{\beta^2},1)\times[0,\frac{1}{\beta})\right)\ .$$

   Let $\mathcal{C}$ be the restriction of the two dimensional Lebesgue σ-algebra on X, and μ the normalized (two dimensional) Lebesgue measure on X. Define on X the transformation $\mathcal{T}_\beta$ as follows :

$$\mathcal{T}_\beta(x,y) := \left(T_\beta x,\ \frac{1}{\beta}(\lfloor \beta x\rfloor + y)\right) \quad \text{for} \quad (x,y)\in X.$$

   (i) Show that $\mathcal{T}_\beta$ is measurable and measure preserving with respect to μ. Prove also that $\mathcal{T}_\beta$ is one-to-one and onto μ a.e.

   (ii) Show that $\mathcal{T}_\beta$ is the natural extension of T_β.

2. Consider $([0,1),\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Lebesgue σ-algebra, and λ is Lebesgue measure. Let $T:[0,1)\to[0,1)$ be defined by

$$Tx = \begin{cases} n(n+1)x - n & \text{if } x\in\left[\frac{1}{n+1},\frac{1}{n}\right) \\ 0 & \text{if } x = 0. \end{cases}$$

Define $a_1 : [0,1)\to[2,\infty]$ by

$$a_1 = a_1(x) = \begin{cases} n+1 & \text{if } x\in\left[\frac{1}{n+1},\frac{1}{n}\right),\ n\ge 1 \\ \infty & \text{if } x=0. \end{cases}$$

For $n\ge 1$, let $a_n = a_n(x) = a_1(T^{n-1}x)$.

(a) Show that T is measure preserving with respect to Lebesgue measure λ.

(b) Show that for λ a.e. x there exists a sequence $a_1, a_2, \cdots$ of positive integers such that $a_i \geq 2$ for all $i \geq 1$, and

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{1}{a_1(a_1-1)\cdots a_{k-1}(a_{k-1}-1)a_k} + \cdots .$$

(c) Consider the dynamical system $(X, \mathcal{F}, \mu, S)$, where $X = \{2, 3, \cdots\}^{\mathbb{N}}$, $\mathcal{F}$ the σ-algebra generated by the cylinder sets, S the left shift on X, and μ the product measure with $\mu(\{x : x_1 = j\}) = \dfrac{1}{j(j-1)}$. Show that $([0,1), \mathcal{B}, \lambda, T)$ and $(X, \mathcal{F}, \mu, S)$ are isomorphic.

3. Use the Shannon-McMillan-Breiman Theorem (and the Ergodic Theorem if necessary) in order to show that

(a) $h_\mu(T) = \log \beta$, where $\beta = \dfrac{1+\sqrt{5}}{2}$, T the β-transformation defined on $([0,1), \mathcal{B})$ by $Tx = \beta x \bmod 1$, and μ the T-invariant measure given by $\mu(B) = \int_B g(x)dx$, where

$$g(x) = \begin{cases} \frac{5+3\sqrt{5}}{10} & 0 \leq x < 1/\beta \\ \frac{5+\sqrt{5}}{10} & 1/\beta \leq x < 1. \end{cases}$$

(b) $h_\mu(T) = -\sum_{j=1}^{m} \sum_{i=1}^{m} \pi_i p_{ij} \log p_{ij}$, where T is the ergodic Markov shift on the space $(\{1, 2, \cdots, m\}^{\mathbb{Z}}, \mathcal{F}, \mu)$, with $\mathcal{F}$ is the σ-algebra generated by the cylinder sets and μ is the Markov measure with stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_m)$ and transition probabilities $(p_{ij} : i, j = 1, \cdots, m)$.

4. Let X be a compact metric space, and $\mathcal{B}$ the Borel σ-algebra on X. Let $T : X \to X$ be a continuous transformation. Let $N \geq 1$ and $x \in X$.

(a) Show that $T^N x = x$ if and only if $\dfrac{1}{N}\displaystyle\sum_{i=0}^{N-1} \delta_{T^i x} \in M(X,T)$. ($\delta_y$ is the Dirac measure concentrated at the point y.)

(b) Suppose $X = \{1, 2, \cdots, N\}$ and $Ti = i+1 \pmod{(N)}$. Show that T is uniquely ergodic. Determine the unique ergodic measure.

5. Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation. Let $k > 0$.

(a) Show that for any finite partition α of X one has $h_\mu(\bigvee_{i=0}^{k-1} \alpha, T^k) = k h_\mu(\alpha, T)$.

(b) Prove that $k h_\mu(T) \leq h_\mu(T^k)$.

(c) Prove that $h_\mu(\alpha, T^k) \leq k h_\mu(\alpha, T)$.

(d) Prove that $h_\mu(T^k) = k h_\mu(T)$.