Exam Measure Theoretic Probability
14 January 2009

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1. Formulate and prove the Monotone Class Theorem.

2. Formulate and prove Hölder's inequality.

3. Let X, Y be a two random variables defined on some $(\Omega, \mathcal{F}, \mathbb{P})$. There always exists a measure μ on $(\mathbb{R}, \mathcal{B})$ such that the laws $\mathbb{P}^X$ and $\mathbb{P}^Y$ of X and Y are both absolutely continuous w.r.t. μ. Denote by f_X and f_Y their Radon-Nikodym derivatives.

   (a) Give an example of such a μ in terms of $\mathbb{P}^X$ and $\mathbb{P}^Y$.

   (b) Assume that X and Y are independent. Show that $\mathbb{P}^{X,Y}$, the law of (X, Y), is absolutely continuous w.r.t. $\mu \times \mu$ and that $\frac{\mathrm{d}\mathbb{P}^{X,Y}}{\mathrm{d}(\mu\times\mu)}(x, y) = f_X(x)f_Y(y)$.

   (c) In general $\mathbb{P}^{X,Y}$ is not absolutely continuous w.r.t. the product measure $\mu \times \mu$. Give a counter example, where μ is Lebesgue measure.

   (d) Assume that $\frac{\mathrm{d}\mathbb{P}^{X,Y}}{\mathrm{d}(\mu\times\mu)}(x, y) = f_X(x)f_Y(y)$. Show that X and Y are independent random variables.

4. Consider a family of Poisson(λ) distributions, for $\lambda \in \Lambda \subset \mathbb{R}^+$. Write ϕ_λ for the corresponding characteristic functions.

   (a) Compute $\phi_\lambda(u)$ for $u \in \mathbb{R}$.

   (b) Show that tightness of the family of Poisson(λ) distributions ($\lambda \in \Lambda$) implies that Λ is bounded.

   (c) Show that this family is tight, if Λ is a bounded set.

   Let $\Lambda = (0, \infty)$ and X have a Poisson(λ) distribution and put $Z_\lambda = \lambda^{-1/2}(X - \lambda)$.

   (d) Find the characteristic function of Z_λ, call it ψ_λ and compute $\lim_{\lambda\to\infty} \psi_\lambda(u)$.

   (e) If $\mathbb{P}^\lambda$ is the distribution of Z_λ, what is the weak limit of the $\mathbb{P}^\lambda$ for $\lambda \to \infty$.

5. Let $X_1, X_2, \dots$ be an *iid* sequence of Bernoulli random variables with $\mathbb{P}(X_k = 0) = 1 - \mathbb{P}(X_k = 1) = 1 - p$, $p \in [0, 1]$. Let $M_n = \sum_{k=1}^{n}(X_k - p)$ and let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

   (a) Show that (M_n) is a martingale.

   Let $\tilde{M}$ be another martingale adapted to the filtration and $m = \mathbb{E}\,\tilde{M}_1$. Since $\tilde{M}$ is adapted, it is known that there exist functions f_n such that $\tilde{M}_n = f_n(X_1, \dots, X_n)$.

   (b) Show that the f_n obey the backward recursion (for $x_i \in \{0, 1\}$, $i \geq 1$)

   $$f_n(x_1, \dots, x_n) = (1 - p) f_{n+1}(x_1, \dots, x_n, 0) + p f_{n+1}(x_1, \dots, x_n, 1).$$

   (c) Let $Y_n = f_n(X_1, \dots, X_{n-1}, 1) - f_n(X_1, \dots, X_{n-1}, 0)$. Show that

   $$\tilde{M}_n = m + (Y \cdot M)_n,$$

   meaning $\tilde{M}_n = m + \sum_{k=1}^{n} Y_k(X_k - p)$.

   (d) Let $\tilde{M}_n = (pe + 1 - p)^{-n} \exp(\sum_{i=1}^{n} X_i)$, $n \geq 0$. Show that $\tilde{M}$ is martingale and that

   $$Y_n = \frac{e - 1}{pe + 1 - p} \tilde{M}_{n-1}.$$