Exam "Wave Attractors"

24 June 2019, 13:30-16:30

No books or lecture notes allowed. **Computations can all use rounded estimates**. *There are 18 subquestions (10 min/question). Weight of question is indicated in points (pt) - 32 points in total.*

1 To mimic certain phenomena in the ocean, a tank experiment is devised on a table that rotates with angular velocity Ω. In that experimental set-up, fluid motions are believed to be described by equations governing the evolution of velocity vector (u, v, w) and reduced pressure p (a perturbation of the hydrostatic pressure), given by:

$$\begin{aligned} u_t - 2\Omega v &= -p_x \\ v_t + 2\Omega u &= -p_y \\ w_t &= -p_z \\ u_x + v_y + w_z &= 0, \end{aligned}$$

where subscript-derivative notation is used.

1a (2 pt) Give at least 4 assumptions used to obtain these equations from the continuity and Navier-Stokes equations[1].

1b (2 pt) Assume the presence of a plane, monochromatic wave $\propto \exp[i(kx + mz - \omega t)]$, propagating in the x, z-plane. What condition do free wave solutions need to satisfy?

[1]In a non-rotating frame, the continuity and Navier-Stokes equations are given by

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u} \tag{2}$$

1c (2 pt) Compute velocity vectors and show that they follow circular trajectories in planes that are perpendicular to the phase velocity vector.
Hint: write down explicit, real-valued expressions for u, v, w, p. The reduced pressure is helpful in the next subquestion.

1d (1 pt) Derive an energy conservation equation for these waves by defining and computing the energy density, E, and energy flux, $\mathbf{F}$.

1e (1pt) Compute the energy flux averaged over one wave period (indicated by a bar) and show it satisfies $\bar{\mathbf{F}} \cdot \mathbf{k} = \mathbf{0}$.

1f (2pt) Assume the fluid is confined to a channel along the x-direction, and that its maximum depth H is small compared to the width of the channel, L. Use scaling to show that the vertical profile of the pressure is determined by the hydrostatic pressure only, i.e. use a perturbation expansion in the aspect ratio, H/L, to show that the equations governing the lowest order fields are given by

$$\begin{aligned} u_t - v &= -p_x \\ v_t + u &= -p_y \\ 0 &= -p_z \\ u_x + v_y + w_z &= 0, \end{aligned}$$

1g (4pt) We will use these equations assuming that the surface $z = 0$ is rigid, the channel is located between $y = 0$ and $y = 1$, and the bottom $z = -h(y)$, is decreasing exponentially, $h = \exp(-sy)$. Here $s > 0$ indicates a constant slope parameter. Searching for a plane monochromatic wave that propagates down-channel, show that the equation governing a suitably defined stream function field $\psi = \Psi(y) \exp i(kx - \omega t)$ is given by

$$\Psi_{yy} + s\Psi_y - (k^2 + \frac{ks}{\omega})\Psi = 0. \tag{3}$$

1h (1pt) What boundary conditions does $\Psi(y)$ need to satisfy?

1i (1pt) Find one (or more) solutions for Ψ satisfying these boundary conditions.

1j (1pt) Discuss any possible constraints on the wave propagation that follow from the solution and boundary condition.

1k (2pt) Discuss whether this wave solution gives a complete view of the free waves that can propagate in a rotating channel filled with a homogeneous-density fluid.

2 We model the dynamics of internal gravity waves in a trench, filled with a non-rotating, uniformly-stratified fluid, $N \approx 4 \times 10^{-4} rad/s$, in the transverse x, z plane. The trench has a vertical wall at $x = 0$, a flat bottom of depth $H = 8km$ over half its width, and a parabolically-curved slope. The total width of the trench spans $L = 24km$. The bottom and slope are given in dimensionless description by

$$z = -\tau, \quad 0 \leq x \leq 1/2,$$
$$z = 4\tau x(x-1), \quad 1/2 \leq x \leq 1.$$

The vertical coordinate is stretched relative to the horizontal, such that the dimensionless depth,

$$\tau = \frac{H}{L}\left(\frac{N^2}{\omega^2} - 1\right)^{1/2}. \tag{4}$$

For internal gravity wave of frequency ω, this stretching ensures that an internal wave beam propagates its energy under an angle of 45 degrees with the horizontal.

2a (3pt) Make a sketch of the model configuration and, as a function of dimensionless depth τ, locate all critical points at the boundary (where the slope is critical, or abruptly changing from subcritical to supercritical).

2b (1pt) Compute the dimensionless depth and frequency intervals for which we can expect to find 1-1 wave attractors, defined as having one reflection at the top and one at the sloping side wall.

2c (2pt) Estimate dimensionless depth τ for the semidiurnal tidal frequency $\omega = 1.4 \times 10^{-4} rad\ s^{-1}$ and check whether it sits in the 1-1 attractor interval. Discuss the likelihood that the semidiurnal tide can manifest itself as such an attractor.

2d (1pt) Use the critical point locations to graphically construct fundamental intervals for the 1-1 attractor.

2e (2pt) Using the critical points, compute the fundamental intervals for $\tau = 3/4$.

2f (2pt) Compute the attractor location for $\tau = 3/4$.

2g (2pt) Discuss qualitatively how to construct internal waves that are forced in a fundamental interval.