Exam "Wave Attractors"

24 June 2019, 13:30-16:30

No books or lecture notes allowed. **Computations can all use rounded estimates**. *There are 18 subquestions (10 min/question). Weight of question is indicated in points (pt) - 32 points in total.*

1 To mimic certain phenomena in the ocean, a tank experiment is devised on a table that rotates with angular velocity Ω. In that experimental set-up, fluid motions are believed to be described by equations governing the evolution of velocity vector (u, v, w) and reduced pressure p (a perturbation of the hydrostatic pressure), given by:

$$u_t - 2\Omega v = -p_x$$
$$v_t + 2\Omega u = -p_y$$
$$w_t = -p_z$$
$$u_x + v_y + w_z = 0,$$

where subscript-derivative notation is used.

1a (2 pt) Give at least 4 assumptions used to obtain these equations from the continuity and Navier-Stokes equations[1].

(1) constant density, (2) transformation to a co-rotating frame of reference, rotating with uniform background rotation, $\Omega = \textit{constant}$, (3) small-amplitude perturbations, allowing linearisation (neglect of nonlinear terms), (4) use of reduced pressure (absorbing gravity and

[1] In a non-rotating frame, the continuity and Navier-Stokes equations are given by

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u} \tag{2}$$

centrifugal force, (5) steady background rotation, hence neglect of Euler force $\propto d\Omega/dt$, (6) (ideal) inviscid fluid, (7) incompressible flow (filtering out sound waves).

1b (2 pt) Assume the presence of a plane, monochromatic wave $\propto \exp[i(kx + mz - \omega t)]$, propagating in the x, z-plane. What condition do free wave solutions need to satisfy?

Inserting the ansatz into the equations, $(u, v, w, p) = (U, V, W, P) \exp[i(kx + mz - \omega t)]$, which has $\partial_y = 0$, one obtains

$$
\begin{aligned}
U - iVf/\omega &= Pk/\omega \\
V &= -iUf/\omega \\
P &= W\omega/m \\
W &= -Uk/m,
\end{aligned}
$$

which, with $f \equiv 2\Omega$, combines into

$$\frac{f^2}{\omega^2} - 1 - \frac{k^2}{m^2} = 0, \tag{3}$$

leading to the dispersion relation, a relation between wave frequency and wave vector,

$$\omega^2 = \frac{f^2 m^2}{k^2 + m^2}. \tag{4}$$

1c (2 pt) Compute velocity vectors and show that they follow circular trajectories in planes that are perpendicular to the phase velocity vector.

Hint: write down explicit, real-valued expressions for u, v, w, p. The reduced pressure is helpful in the next subquestion.

Using subscript r to denote the real part, e.g. $u_r \equiv \Re(u)$, we obtain

$$
\begin{aligned}
u_r &= U \cos(kx + mz - \omega t), \\
v_r &= U \frac{f}{\omega} \sin(kx + mz - \omega t), \\
w_r &= -\frac{k}{m} U \cos(kx + mz - \omega t), \\
p_r &= -\frac{\omega k}{m^2} U \cos(kx + mz - \omega t).
\end{aligned}
$$

This implies u_r, w_r, p_r are in phase, and v_r is out-of-phase. This implies that fluid particles move simultaneously in x and z direction at a velocity magnitude $U\sqrt{1+k^2/m^2} = Uf/\omega$, which is identical to the amplitude of v_r, although v_r lags by a quarter cycle. Hence the velocity vector traces a circular motion inclined under a slope $-1/s$ making an angle $\arctan(-k/m) = \pi/2 - \alpha$ with the horizontal, where slope $s = \tan\alpha = m/k$ is the inclination of the wave vector $\mathbf{k} = (\mathbf{k}, \mathbf{0}, \mathbf{m})$ with the horizontal, verifying that $\mathbf{u} \cdot \mathbf{k} = \mathbf{0}$.

1d (1 pt) Derive an energy conservation equation for these waves by defining and computing the energy density, E, and energy flux, $\mathbf{F}$.

Multiply the momentum equation by $\mathbf{u}$ and the continuity equation by p and add. This yields

$$E_t + \nabla \cdot \mathbf{F} = \mathbf{0}, \tag{5}$$

where $E \equiv \frac{1}{2}(u^2 + v^2 + w^2)$ and $\mathbf{F} = \mathbf{pu}$.

1e (1pt) Compute the energy flux averaged over one wave period (indicated by a bar) and show it satisfies $\bar{\mathbf{F}} \cdot \mathbf{k} = \mathbf{0}$.

Inserting the real expressions of $\mathbf{u_r}$ and p_r into $\mathbf{F}$, upon averaging over one period we find

$$\bar{\mathbf{F}} = \frac{\omega k U^2}{2m^3}(-m, 0, k), \tag{6}$$

indeed directed along the velocity vector direction and perpendicular to $\mathbf{k}$.

1f (2pt) Assume the fluid is confined to a channel along the x-direction, and that its maximum depth H is small compared to the width of the channel, L. Use scaling to show that the vertical profile of the pressure is determined by the hydrostatic pressure only, i.e. use a perturbation expansion in the aspect ratio, H/L, to show that the equations governing the lowest order fields are given by

$$u_t - v = -p_x$$
$$v_t + u = -p_y$$
$$0 = -p_z$$
$$u_x + v_y + w_z = 0,$$

Replace the original variables by the following dimensionless and scaled (primed) variables $(x, y) = L(x', y')$, $z = Hz'$, $t = t'/f$, $(u, v) = Lf(u', v')$, $w = Hfw'$, $p = L^2 f^2 p'$. Subsequently, divide the horizontal momentum equations by Lf^2, the vertical momentum equation by $L^2 f^2/H$. Then, a small parameter $\delta \equiv (H^2/L^2)$ appears at a single spot, namely multiplying the vertical acceleration term. Inserting perturbations expansions $u' = u_0 + \delta u_1 + \delta^2 u_2 + \cdots$, and similarly for all other variables, and ordering terms of the same power in δ (assuming all fields $u_0, u_1, u_2, \cdots$ are of $O(1)$), to lowest order we receive the given approximate equations (upon dropping the subscript 0.) This implies that lowest-order reduced pressure p_0 (here p) is independent of depth z, and, hence is determined by surface pressure (due, e.g. to surface waves or atmospheric pressure) only.

1g (4pt) We will use these equations assuming that the surface $z = 0$ is rigid, the channel is located between $y = 0$ and $y = 1$, and the bottom $z = -h(y)$, is decreasing exponentially, $h = \exp(-sy)$. Here $s > 0$ indicates a constant slope parameter. Searching for a plane monochromatic wave that propagates down-channel, show that the equation governing a suitably defined stream function field $\psi = \Psi(y) \exp i(kx - \omega t)$ is given by

$$\Psi_{yy} + s\Psi_y - (k^2 + \frac{ks}{\omega})\Psi = 0. \tag{7}$$

Since p is z-independent, so are u, v and horizontal divergence $u_x + v_y$. Therefore w must be linear in z. In order to comply with the surface $w = 0$ and bottom $w = -vh_y$ boundary conditions

$$w = \frac{z}{h} v h_y. \tag{8}$$

This implies that $w_z = vh_y/h$, so that the horizontal transport vector $h(u, v)$ must be non-divergent $(hu)_x + (hv)_y = 0$. Therefore a transport streamfunction ψ can be defined through

$hu = -\psi_y, \;\; hv = \psi_x$. Inserting this in the vorticity ($q = v_x - u_y$) equation, $q_+ v h_y/h = 0$t hat we can extract by cross-differentiation from the horizontal momentum equations, we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{h}(\psi_{xx} + \psi_{yy} + (\frac{1}{h})_y \psi_y\right) + \frac{h_y \psi_x}{h^2} = 0. \tag{9}$$

Now using the plane wave ansatz, $\psi = \Psi(y) \exp i(kx - \omega t)$, using $h_y = -sh$, yields

$$\Psi_{yy} + s\Psi_y - (k^2 + \frac{ks}{\omega})\Psi = 0, \tag{10}$$

as desired.

1h (1pt) What boundary conditions does $\Psi(y)$ need to satisfy?

As there can be no flow through the boundaries of the channel, $\Psi = 0$ at $y = 0, 1$.

1i (1pt) Find one (or more) solutions for Ψ satisfying these boundary conditions.

Since there is no explicit y-dependence, we can insert a trial solution $\Psi = \exp(\lambda y)$, and require $\lambda^2 + s\lambda - k^- \frac{ks}{\omega} = 0$. This is solved by $\lambda_\pm = -\frac{s}{2} \pm \sqrt{\frac{s^2}{4} + k^2 + \frac{sk}{\omega}}$. This leads to exponentially growing or decaying solutions that are unable to satisfy vanishing streamfunction at both $y = 0$ and $y = 1$ unless the argument of the square root is negative. Defining therefore $l^2 = -(\frac{s^2}{4} + k^2 + \frac{sk}{\omega})$, we find solutions

$$\psi = \hat{\psi} \exp[-sy/2 + i(kx - \omega t)] \sin(ly), \tag{11}$$

vanishing at the boundaries when $l = n\pi, \;\; n \in \mathbf{N}$. (The $\cos(ly)$ are ruled out by the boundary conditions.)

1j (1pt) Discuss any possible constraints on the wave propagation that follow from the solution and boundary condition.

The expression for l^2 can be rewritten in terms of frequency as

$$\omega = \frac{-sk}{k^2 + l^2 + s^2/4}. \tag{12}$$

This shows that as $\omega > 0$ by convention, for $s > 0$ the wave needs to have $k < 0$, hence propagate its phase with the shallow side on its right, looking into the direction of propagation (i.e. into negative x direction).

1k (2pt) Discuss whether this wave solution gives a complete view of the free waves that can propagate in a rotating channel filled with a homogeneous-density fluid.

No. First of all it is just the lowest order term of an in principle infinite set of terms. Who knows how the summed series sets up the impression we get from the lowest order field... Second, the exact equations exhibit inertial waves, which, in the presence of a sloping bottom, should exhibit wave focusing onto an attractor.

2 We model the dynamics of internal gravity waves in a trench, filled with a non-rotating, uniformly-stratified fluid, $N \approx 4 \times 10^{-4} rad/s$, in the transverse x, z plane. The trench has a vertical wall at $x = 0$, a flat bottom of depth $H = 8km$ over half its width, and a parabolically-curved slope. The total width of the trench spans $L = 24km$. The bottom and slope are given in dimensionless description by

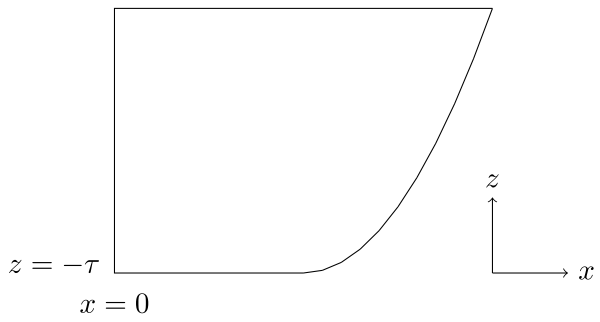
$$\begin{aligned} z &= -\tau, \quad 0 \leq x \leq 1/2, \\ z &= 4\tau x(x-1), \quad 1/2 \leq x \leq 1. \end{aligned}$$

The vertical coordinate is stretched relative to the horizontal, such that the dimensionless depth,

$$\tau = \frac{H}{L}\left(\frac{N^2}{\omega^2} - 1\right)^{1/2}. \tag{13}$$

For internal gravity wave of frequency ω, this stretching ensures that an internal wave beam propagates its energy under an angle of 45 degrees with the horizontal.

2a (3pt) Make a sketch of the model configuration and, as a function of dimensionless depth τ, locate all critical points at the boundary (where the slope is critical, or abruptly changing from subcritical to supercritical).



Critical points exist in the three acute corners (which have 'all' inclinations) and in one point on the parabolic bottom where its slope equals 1, namely at $(x_c, z_c) = (\frac{1}{2}(1+\frac{1}{4\tau}), -\tau(1-\frac{1}{16\tau^2})$, provided $x_c > 0$, i.e. provided $\tau > \frac{1}{4}$. For smaller τ-values the bottom is everywhere subcritical and the corner $(1,0)$ attracts. In that case there are only 3 critical points (the corners).

2b (1pt) Compute the dimensionless depth and frequency intervals for which we can expect to find 1-1 wave attractors, defined as having one reflection at the top and one at the sloping side wall.

1-1 attractors are found in between two degenerate cases, namely (i) when (1,0) connects to $(0,-\tau)$; this happens for $\tau = 1$, and (ii) when (0,0) connects to the critical point $(x_c, -x_c)$, which happens when $z_c = -\tau(1-\frac{1}{4\tau})(1+\frac{1}{4\tau}) = -x_c = -\frac{1}{2}(1+\frac{1}{4\tau})$. Dividing out the common factor shows this happens for $\tau = 3/4$. Hence for $3/4 \leq \tau \leq 1$ we have 1-1 attractors. In terms of frequency we have to convert this from τ's definition: for $\frac{N}{\sqrt{(L/H)^2+1}} \leq \omega \leq \frac{N}{\sqrt{(3L/4H)^2+1}}$. Given the values of N, H, L, we estimate: $1.33 \times 10^{-4} \leq \omega \leq 1.6 \times 10^{-4}$

2c (2pt) Estimate dimensionless depth τ for the semidiurnal tidal frequency $\omega = 1.4 \times 10^{-4} rad\ s^{-1}$ and check whether it sits in the 1-1 attractor interval. Discuss the likelihood that the semidiurnal tide can manifest itself as such an attractor.

The previously estimated interval shows it is likely that the semidiurnal tide sits in the 1-1 attractor band. Indeed, $\tau = \frac{8}{24}\left((\frac{4}{1.4})^2 - 1\right)^{1/2} \approx \sqrt{7}/3 \approx 0.9 < 1$ also confirms it sits in the 1-1 τ interval.

2d (1pt) Use the critical point locations to graphically construct fundamental intervals for the 1-1 attractor.

Take a typical value for τ relatively large. Then find FIs by mapping the lower left corner up to the surface to $(\tau, 0)$, which gives the left boundary of the segment whose right boundary is at (0,1). The second one should be launched from the critical point on the parabola, upward, and its intersection with the left wall, gives a vertical FI between that intersection and the origin.

2e (2pt) Using the critical points, compute the fundamental intervals for $\tau = 3/4$.

For $\tau = 3/4$, the critical location is at $(x_c, z_c) = (2/3, -2/3)$. Then fundamental intervals (FIs) are obtained by mapping one critical point $(0, -\tau)$, say, along the characteristic $z = x - \tau$ up to $z = 0$ and find $x_s = \tau = 3/4$. So the interval $3/4 \leq x \leq 1$ at $z = 0$ serves e.g. as the first FI. The second is obtained by mapping the critical point at the bottom along $z = -x + x_c + z_c$ to the surface, $z = 0$. But we find $x_s = x_c + z_c = 0$. Hence, for $\tau = 3/4$ we therefore find that the surface FI shrinks to the corner point (0,0).

2f (2pt) Compute the attractor location for $\tau = 3/4$.

It is the degenerate line $z = -x$ for $0 \leq x \leq 2/3$.

2g (2pt) Discuss qualitatively how to construct internal waves that are forced in a fundamental interval.

Prescribe partial pressure on the fundamental interval, which is half the pressure at that interval. It is invariant along the webs of characteristics launched from that interval when partial pressure is taken real. In that case streamfunction is zero at that interval, so it prescribes a free, standing (sloshing) internal gravity wave (at least over the region reached by *that* FI. The complete pressure and streamfunction fields are also determined by prescribed partial pressures in the other FIs.

If the partial pressure is taken complex, but still the same over the whole webs, one constructs a free propagating internal wave field.

If however, the partial pressure is taken invariant over the two *half-webs* emanating from the FI, internal waves propagate into the fluid domain (and towards the attractor).