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Brownian Motion and Financial Mathematics: Final 2015-16

(1) Let $\{W(t) : t \geq 0\}$ be Brownian motion. Compute $\int_0^t \sin(W(s))\, dW(s)$, and express your answer in the form $\int_0^t h(W(s))\, ds + g(W(t))$, for explicit deterministic functions h and g. (1.5 pts)

(2) The evolution of a stock price $S(t)$ is modeled by

$$S(t) = e^{\mu t + \sigma W(t)},$$

where $W(t)$ is a standard Brownian motion with filtration $\{\mathcal{F}(t) : t \geq 0\}$, and μ and $\sigma > 0$ are real parameters. Assume that the initial value of the stock is $S(0) = 1$.

(a) Determine an expression for $P(S(t) \leq x)$, for $x \geq 0$. (0.5 pts)

(b) Derive expressions for the median, and expectation of $S(t)$. Note that the median is the value m such that $P(S(t) \leq m) = 1/2$. (1 pt)

(c) Determine an expression for the conditional expectation $E[S(t) \,|\, \mathcal{F}(s)]$ with $s < t$. Find conditions on μ and σ under which the price process $\{S(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. (1 pt)

(3) Let $\{B_1(t) : t \geq 0\}$ and $\{B_2(t) : t \geq 0\}$ be a pair of correlated Brownian motions with

$$dB_1(t)dB_2(t) = \rho(t)dt,$$

with $\{\rho(t) : t \geq 0\}$ a stochastic process taking values in $[-1, 1]$ which is adapted to the filtration $\{\mathcal{F}(t) : t \geq 0\}$ generated by the Brownian motions $B_1(t)$ and $B_2(t)$. Define two processes $W_1(t)$ and $W_2(t)$ by

$$dW_1(t) = dB_1(t),$$

and

$$dW_2(t) = \alpha(t)dB_1(t) + \beta(t)dB_2(t),$$

with $\{\alpha(t) : t \geq 0\}$ and $\{\beta(t) : t \geq 0\}$ adapted processes, and $\beta(t) \geq 0$ for $t \geq 0$. Find the values of $\alpha(t)$, $\beta(t)$ such that the random process $\{(W_1(t), W_2(t)) : t \geq 0\}$ is a 2-dimensional Brownian motion. (2 pts)

(4) Suppose that the stock price $S(t)$ is a geometric Browninan motion, i.e.

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW(t),$$

where $W(t)$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}(t) : t \geq 0\}$. Let r be the interest rate, and $\theta = \dfrac{\alpha - r}{\sigma}$. Consider the process

$$Z(t) = e^{-\theta W(t) - (r + \frac{1}{2}\theta^2)t}.$$

(a) Show that

$$dZ(t) = -\theta Z(t)\, dW(t) - rZ(t)\, dt.$$

(0.5 pts)

(b) Consider the portfolio process $X(t) = \Delta(t)S(t) + (X(t) - \Delta(t)S(t))$. Show that $\{Z(t)X(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. (1 pt)

(c) Let $T > 0$ be a fixed terminal time, and assume $\mathcal{F} = \mathcal{F}(T)$. Let $V(T)$ be an $\mathcal{F}(T)$-measurable function (thought of as the payoff of a derivative with expiration date T). Show that if an investor wants to begin with some initial value $X(0)$ and invests in order to have a portfolio with value $V(T)$ at time T, then he must begin with initial capital $X(0) = E[Z(T)V(T)]$. (0.5 pts)

(5) Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be the filtration generated by the Brownian motion. Let $\{\Theta(t) : 0 \leq t \leq T\}$ be a bounded adapted process. Use Girsanov's Theorem as well as the Martingale Representation Theorem to show that if Y is an $\mathcal{F}(T)$ measurable function, then there exist a constant x and an adapted process $\{\alpha(t) : 0 \leq t \leq T\}$ such that

$$Y = x + \int_0^T \alpha(t)\Theta(t)dt + \int_0^T \alpha(t)dW(t).$$

(2pts)