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**Brownian Motion and Financial Mathematics: Solution Final 2015-16**

(1) Let $\{W(t) : t \geq 0\}$ be Brownian motion. Compute $\int_0^t \sin(W(s))\, dW(s)$, and express your answer in the form $\int_0^t h(W(s))\, ds + g(W(t))$, for explicit deterministic functions $h$ and $g$. (1.5 pts)

**Proof**: Let $f(x) = \cos(x)$, then $f$ is continuously differentiable, with $f'(x) = -\sin(x)$ and $f''(x) = -\cos(x)$. By Ito's Lemma, we have

$$\cos(W(t)) = \cos(W(0)) - \int_0^t \sin(W(s))\, dW(s) - \int_0^t \frac{1}{2}\cos(W(s))\, ds.$$

Since $\cos(W(0)) = 1$, we have

$$\int_0^t \sin(W(s))\, dW(s) = 1 - \cos(W(t)) - \int_0^t \frac{1}{2}\cos(W(s))\, ds.$$

The result follows with $f(x) = 1 - \cos(x)$ and $g(x) = -\cos(x)$.

(2) The evolution of a stock price $S(t)$ is modeled by

$$S(t) = e^{\mu t + \sigma W(t)},$$

where $W(t)$ is a standard Brownian motion with filtration $\{\mathcal{F}(t) : t \geq 0\}$, and $\mu$ and $\sigma > 0$ are real parameters. Assume that the initial value of the stock is $S(0) = 1$.

(a) Determine an expression for $P(S(t) \leq x)$, for $x \geq 0$. (0.5 pts)

(b) Derive expressions for the median, and expectation of $S(t)$. Note that the median is the value $m$ such that $P(S(t) \leq m) = 1/2$. (1 pt)

(c) Determine an expression for the conditional expectation $E[S(t) \,|\, \mathcal{F}(s)]$ with $s < t$. Find conditions on $\mu$ and $\sigma$ under which the price process $\{S(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. (1 pt)

**Proof (a)**: Note that $W(t)$ is normally distributed with mean 0 and variance $t$, hence the random variable $Z(t) = \dfrac{W(t)}{\sqrt{t}}$ is standard normal. Now,

$$P(S(t) \leq x) = P(e^{\mu t + \sigma W(t)} \leq x) = P(W(t) \leq \frac{\ln x - \mu t}{\sigma}) = P(Z(t) \leq \frac{\ln x - \mu t}{\sigma\sqrt{t}}) = N(\frac{\ln x - \mu t}{\sigma\sqrt{t}}),$$

where $N$ denotes the standard normal distribution function.

**Proof (b)**: We are looking for the value of $m$ such that

$$P(S(t) \leq m) = P(Z(t) \leq \frac{\ln m - \mu t}{\sigma\sqrt{t}}) = 1/2.$$

Since $Z(t)$ is a symmetric standard normal random variable, the median of $Z(t)$ is 0. This implies that $\dfrac{\ln m - \mu t}{\sigma\sqrt{t}} = 0$ leading to $m = e^{\mu t}$. To calculate the expectation of $S(t)$, we first note that the moment generating function of the standard normal random variable $Z(t)$ satisfies $E(e^{sZ(t)}) = e^{\frac{1}{2}s^2}$. Now

$$E(S(t)) = E(e^{\mu t + \sigma W(t)}) = e^{\mu t} E(e^{\sigma\sqrt{t}Z(t)}) = e^{\mu t} e^{\frac{1}{2}\sigma^2 t} = e^{t(\mu + \frac{1}{2}\sigma^2)}.$$

**Proof (c)**: For $s < t$ we have, $S(t) = S(s)e^{\mu(t-s)+\sigma(W(t)-W(s))}$. Furthermore, $S(s)$ is $\mathcal{F}(s)$-measurable, and $W(t) - W(s)$ is independent of $\mathcal{F}(s)$. Thus,

$$E(S(t)\,|\,\mathcal{F}(s)) = S(s)e^{\mu(t-s)}E(e^{\sigma(W(t)-W(s))}) = S(s)e^{(\mu+\frac{1}{2}\sigma^2)(t-s)},$$

the last equality follows from the moment generating function of the normally distributed random variable $W(t)-W(s)$. For the process $S(t)$ to be a martingale, we must have $E(S(t)\,|\,\mathcal{F}(s)) = S(s)$ for $s < t$. This leads to $e^{(\mu+\frac{1}{2}\sigma^2)(t-s)} = 1$ or equivalently, $\mu = -\frac{1}{2}\sigma^2$.

(3) Let $\{B_1(t) : t \geq 0\}$ and $\{B_2(t) : t \geq 0\}$ be a pair of correlated Brownian motions with

$$dB_1(t)dB_2(t) = \rho(t)dt,$$

with $\{\rho(t) : t \geq 0\}$ a stochastic process taking values in $[-1, 1]$ which is adapted to the filtration $\{\mathcal{F}(t) : t \geq 0\}$ generated by the Brownian motions $B_1(t)$ and $B_2(t)$. Define two processes $W_1(t)$ and $W_2(t)$ by

$$dW_1(t) = dB_1(t),$$

and

$$dW_2(t) = \alpha(t)dB_1(t) + \beta(t)dB_2(t),$$

with $\{\alpha(t) : t \geq 0\}$ and $\{\beta(t) : t \geq 0\}$ adapted processes, and $\beta(t) \geq 0$ for $t \geq 0$. Find the values of $\alpha(t)$, $\beta(t)$ such that the random process $\{(W_1(t), W_2(t)) : t \geq 0\}$ is a 2-dimensional Brownian motion. (2 pts)

**Proof**: We use Levy characterization of a two-dimensional Brownian motion (Theorem 4.6.5). By definition of $W_1(t)$ and $W_2(t)$, it follows directly that $W_1(t)$ is a Brownian motion and $W_2(t)$ is a martingale with continuous paths and $W_2(0) = 0$. It remains to find the values of $\alpha(t)$ and $\beta(t)$ such that the quadratic variation of $W_2(t)$ is $t$ which will imply that $\{W_2(t) : t \geq 0\}$ is a Brownian motion, and the cross-variation of $W_1(t)$ and $W_2(t)$ is zero which implies that $W_1(t)$ and $W_2(t)$ are independent. Thus, we want

$$dW_2(t)dW_2(t) = (\alpha^2(t) + \beta^2(t) + 2\rho(t)\alpha(t)\beta(t))dt = dt,$$

and

$$dW_1(t)dW_2(t) = (\alpha(t) + \beta(t)\rho(t))dt = 0.$$

These lead to the system

$$\alpha^2(t) + \beta^2(t) + 2\rho(t)\alpha(t)\beta(t) = 1,$$

and

$$\alpha(t) + \beta(t)\rho(t) = 0.$$

Solving we get $\beta(t) = \dfrac{1}{\sqrt{1-\rho^2(t)}}$, and $\alpha(t) = \dfrac{-\rho(t)}{\sqrt{1-\rho^2(t)}}$.

(4) Suppose that the stock price $S(t)$ is a geometric Browninan motion, i.e.

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t),$$

where $W(t)$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}(t) : t \geq 0\}$. Let $r$ be the interest rate, and $\theta = \dfrac{\alpha - r}{\sigma}$. Consider the process

$$Z(t) = e^{-\theta W(t) - (r+\frac{1}{2}\theta^2)t}.$$

(a) Show that

$$dZ(t) = -\theta Z(t)\,dW(t) - rZ(t)\,dt.$$

(0.5 pts)

(b) Consider the portfolio process $X(t) = \Delta(t)S(t) + (X(t) - \Delta(t)S(t))$. Show that $\{Z(t)X(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. (1 pt)

(c) Let $T > 0$ be a fixed terminal time, and assume $\mathcal{F} = \mathcal{F}(T)$. Let $V(T)$ be an $\mathcal{F}(T)$-measurable function (thought of as the payoff of a derivative with expiration date $T$). Show that if an investor wants to begin with some initial value $X(0)$ and invests in order to have a portfolio with value $V(T)$ at time $T$, then he must begin with initial capital $X(0) = E[Z(T)V(T)]$. (0.5 pts)

**Proof (a)**: We apply Ito-Doeblin's formula to the function $f(t,x) = e^{-\theta x - (r+\frac{1}{2}\theta^2)t}$. Note that $f_t(t,x) = -(r+\frac{1}{2}\theta^2)f(t,x)$, $f_x(t,x) = -\theta f(t,x)$ and $f_{xx}(t,x) = \theta^2 f(t,x)$. Now,

$$dZ(t) = d(f(t,W(t)) = -(r+\frac{1}{2}\theta^2)Z(t)dt - \theta Z(t)dW(t) + \frac{1}{2}\theta^2 dt = -rZ(t)dt - \theta Z(t)dW(t).$$

**Proof (b)**: It suffices to show that $Z(t)X(t)$ is an Ito-integral. We know that the underlying stochastic differential equation of $X(t)$ is given by

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))d(t).$$

Replacing $dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$ we get

$$dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t).$$

We now apply Ito's product rule, we have

$$d(Z(t)X(t)) = Z(t)dX(t) + X(t)dZ(t) + dX(t)dZ(t).$$

Using the equalities $dZ(t) = -rZ(t)dt - \theta Z(t)dW(t)$, $dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)$ and $\sigma\theta = \alpha - r$ we get

$$d(Z(t)X(t)) = \Big(\Delta(t)\sigma S(t)Z(t) - \theta X(t)Z(t)\Big)dW(t).$$

This implies that $Z(t)X(t)$ is an Ito-integral and hence a martingale.

**Proof (c)**: Our portfolio should satisfy $X(T) = V(T)$ and hence $Z(T)X(T) = Z(T)V(T)$. By part (b), $\{Z(t)X(t)\}$ is a martingale, hence

$$X(0) = Z(0)X(0) = E(Z(T)X(T)) = E(Z(T)V(T)).$$

(5) Let $\{W(t) : 0 \le t \le T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}(t) : 0 \le t \le T\}$ be the filtration generated by the Brownian motion. Let $\{\Theta(t) : 0 \le t \le T\}$ be a bounded adapted process. Use Girsanov's Theorem as well as the Martingale Representation Theorem to show that if $Y$ is an $\mathcal{F}(T)$ measurable function, then there exist a constant $x$ and an adapted process $\{\alpha(t) : 0 \le t \le T\}$ such that

$$Y = x + \int_0^T \alpha(t)\Theta(t)dt + \int_0^T \alpha(t)dW(t).$$

(2 pts)

**Proof** : Let $Z(t) = e^{-\int_0^T \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du}$, and $\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du$. Consider measure $\widetilde{P}$ satisfying $d\widetilde{P} = Z(T)dP$. Since the process $\{\Theta(t) : 0 \le t \le T\}$ is bounded, then $E\Big(\int_0^T \Theta^2(u)Z^2(u)\,du\Big) < \infty$ as well as $\widetilde{E}\Big(\int_0^T \Theta^2(u)Z^2(u)\,du\Big) < \infty$, by Girsanov's Theorem (in fact Corollary 5.3.2), $\{\widetilde{W}(t) : 0 \le t \le T\}$ is a Brownian motion under the measure $\widetilde{P}$. Now, let $Y$ be an $\mathcal{F}(T)$ measurable function, and define $X(t) = \widetilde{E}(Y \,|\, \mathcal{F}(t))$ for $0 \le t \le T$. Then, $\{X(t) : 0 \le t \le T\}$ is a martingale under the measure $\widetilde{P}$. Since $Y$ is an $\mathcal{F}(T)$ measurable, we have $X(T) = \widetilde{E}(Y \,|\, \mathcal{F}(T)) = Y$. By the Martingale Representation Theorem, there exists an adapted process $\{\alpha(t) : 0 \le t \le T\}$ such that

$$X(t) = X(0) + \int_0^t \alpha(u)d\widetilde{W}(u) = X(0) + \int_0^t \alpha(u)\Theta(u)du + \int_0^t \alpha(u)dW(u),\ 0 \le t \le T.$$

Now set $x = X(0)$, since $Y = X(T)$, we have

$$Y = x + \int_0^T \alpha(t)\Theta(t)dt + \int_0^T \alpha(t)dW(t).$$